Artin-Schreier-Witt 拡大と正規底

諏訪紀幸 (中央大学)

早稲田，２０１２年７月６日

Example 1. The case of $p = 2$, $n = 2$. We have

$$E_2(X, 1; T) \equiv 1 + XT + (X + X^2)T^2 + (X + X^2)T^3 \mod (2, T^4),$$

and therefore

$$E_2(X_0, X_1, 1; T) \equiv 1 + X_0T + (X_0 + X_0^2 + X_1)T^2 + (X_0 + X_0^2 + X_0X_1)T^3 \mod (2, T^4).$$

Hence we obtain

$$\begin{aligned}
\Psi_0(X_0, X_1) &= 1 + X_0 + X_1 + X_0X_1 = (1 + X_0)(1 + X_1),\\
\Psi_1(X_0, X_1) &= X_0^2 + X_0X_1 = X_0(X_0 + X_1),\\
\Psi_2(X_0, X_1) &= X_1 + X_0X_1 = (1 + X_0)X_1,\\
\Psi_3(X_0, X_1) &= X_0 + X_0^2 + X_0X_1 = X_0(1 + X_0 + X_1).
\end{aligned}$$

On the other hand, the endomorphism

$$F - 1 : W_{2,\mathbb{F}_2} = \operatorname{Spec} \mathbb{F}_2[X_0, X_1] \to W_{2,\mathbb{F}_2} = \operatorname{Spec} \mathbb{F}_2[X_0, X_1]$$

is defined by

$$(X_0, X_1) \mapsto (X_0^2 + X_0, X_1^2 + X_1 + X_0^3 + X_0^2).$$

Let R be an $\mathbb{F}_2$-algebra and $a_0, a_1 \in R$. Put

$$S = R[X_0, X_1]/(X_0^2 + X_0 + a_0, X_1^2 + X_1 + X_0^3 + X_0^2 + a_1),$$

and let α_0 and α_1 denote the image of X_0 and X_1 in S, respectively. Then S/R is an unramified cyclic extension of degree 4. The Galois group of S/R is generared by

$$\gamma : (\alpha_0, \alpha_1) \mapsto (\alpha_0 + 1, \alpha_1 + \alpha_0).$$

Furthermore $\Psi_0(\alpha_0, \alpha_1) = (1 + \alpha_0)(1 + \alpha_1)$ generates a normal basis of S/R.

Example 2. The case of $p = 2$, $n = 3$. We have an equalilty in $\mathbb{F}_2[X, T]/(T^8)$

$$\begin{aligned} E_2(X, 1; T) &= 1 + XT + (X + X^2)T^2 + (X + X^2)T^3 + (X^2 + X^3)T^4 \\ &\quad + (X + X^2 + X^3 + X^5)T^5 + (X + X^2)T^6 + (X + X^3 + X^6 + X^7)T^7, \end{aligned}$$

and therefore an equality in $\mathbb{F}_2[X_0, X_1, X_2, T]/(T^8)$

$$\begin{aligned} &E_2(X_0, X_1, X_2, 1; T) = \\ &\quad 1 + X_0T + (X_0{+}X_0^2{+}X_1)T^2 + (X_0{+}X_0^2{+}X_0X_1)T^3 \\ &\quad + (X_0^2{+}X_0^3{+}X_1{+}X_0X_1{+}X_0^2X_1{+}X_1^2{+}X_2)T^4 \\ &\quad + (X_0{+}X_0^2{+}X_0^3{+}X_0^5{+}X_0^2X_1{+}X_0X_1^2{+}X_0X_2)T^5 \\ &\quad + (X_0{+}X_0^2{+}X_1{+}X_0X_1{+}X_0^3X_1{+}X_1^2{+}X_0X_1^2{+}X_0^2X_1^2{+}X_0X_2{+}X_0^2X_2{+}X_1X_2)T^6 \\ &\quad + (X_0{+}X_0^3{+}X_0^6{+}X_0^7{+}X_0X_1{+}X_0^3X_1{+}X_0^5X_1{+}X_0^2X_1^2{+}X_0X_2{+}X_0^2X_2{+}X_0X_1X_2)T^7. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \Psi_0(\boldsymbol{X}) &= 1{+}X_0^2{+}X_0^3{+}X_0^5{+}X_0^6{+}X_0^7{+}X_1{+}X_0^5X_1{+}X_2{+}X_0X_2{+}X_1X_2{+}X_0X_1X_2, \\ \Psi_1(\boldsymbol{X}) &= X_0^5{+}X_0^6{+}X_0^7{+}X_0^2X_1{+}X_0^3X_1{+}X_0^5X_1{+}X_0X_1^2{+}X_0^2X_1^2{+}X_0^2X_2{+}X_0X_1X_2, \\ \Psi_2(\boldsymbol{X}) &= X_0^2{+}X_0^3{+}X_0^6{+}X_0^7{+}X_0X_1{+}X_0^5X_1{+}X_1^2{+}X_0X_1^2{+}X_1X_2{+}X_0X_1X_2, \\ \Psi_3(\boldsymbol{X}) &= X_0^2{+}X_0^3{+}X_0^6{+}X_0^7{+}X_0^3X_1{+}X_0^5X_1{+}X_0^2X_1^2{+}X_0X_2{+}X_0^2X_2{+}X_0X_1X_2, \\ \Psi_4(\boldsymbol{X}) &= X_0{+}X_0^2{+}X_0^3{+}X_0^5{+}X_0^6{+}X_0^7{+}X_0X_1{+}X_0^5X_1{+}X_2{+}X_0X_2{+}X_1X_2{+}X_0X_1X_2, \\ \Psi_5(\boldsymbol{X}) &= X_0^2{+}X_0^5{+}X_0^6{+}X_0^7{+}X_0X_1{+}X_0^2X_1{+}X_0^3X_1{+}X_0^5X_1{+}X_0X_1^2{+}X_0^2X_1^2{+}X_0^2X_2{+}X_0X_1X_2, \\ \Psi_6(\boldsymbol{X}) &= X_0^2{+}X_0^3{+}X_0^6{+}X_0^7{+}X_1{+}X_0^5X_1{+}X_1^2{+}X_0X_1^2{+}X_1X_2{+}X_0X_1X_2, \\ \Psi_7(\boldsymbol{X}) &= X_0{+}X_0^3{+}X_0^6{+}X_0^7{+}X_0X_1{+}X_0^3X_1{+}X_0^5X_1{+}X_0^2X_1^2{+}X_0X_2{+}X_0^2X_2{+}X_0X_1X_2. \end{aligned}$$

On the other hand, the endomorphism

$$F - 1 : W_{3,\mathbb{F}_2} = \operatorname{Spec} \mathbb{F}_2[X_0, X_1, X_2] \to W_{3,\mathbb{F}_2} = \operatorname{Spec} \mathbb{F}_2[X_0, X_1, X_2]$$

is defined by

$$(X_0, X_1, X_2) \mapsto (\tilde{F}_0(X_0), \tilde{F}_1(X_0, X_1), \tilde{F}_2(X_0, X_1, X_2)),$$

where

$$\begin{aligned} \tilde{F}_0(X_0) &= X_0^2 + X_0, \\ \tilde{F}_1(X_0, X_1) &= X_1^2 + X_1 + X_0^3 + X_0^2, \\ \tilde{F}_2(X_0, X_1, X_2) &= X_2^2 + X_2 + X_1^3 + X_1^2X_0^3 + X_1^2X_0^2 + X_1^2 + X_1X_0^3 + X_1X_0^2 + X_0^7 + X_0^5. \end{aligned}$$

Let R be an $\mathbb{F}_2$-algebra and $a_0, a_1, a_2 \in R$. Put

$$S = R[X_0, X_1, X_2]/(\tilde{F}_0(X_0) + a_0, \tilde{F}_1(X_0, X_1) + a_1, \tilde{F}_2(X_0, X_1, X_2) + a_2),$$

and let α_0, α_1 and α_2 denote the image of X_0, X_1 and X_2 in S, respectively. Then S/R is an unramified cyclic extension of degree 8. The Galois group of S/R is generared by

$$\gamma : (\alpha_0, \alpha_1, \alpha_2) \mapsto (\alpha_0 + 1, \alpha_1 + \alpha_0, \alpha_2 + \alpha_1\alpha_0 + \alpha_0^3 + \alpha_0).$$

Furthermore $\Psi_0(\alpha_0, \alpha_1, \alpha_2)$ generates a normal basis of S/R.

Example 3. The case of $p = 3$, $n = 2$. We have an equality in $\mathbb{F}_3[X, T]/(T^9)$

$$\begin{aligned} E_3(X, 1; T) = 1 &+ XT + (X + 2X^2)T^2 + (X^2 + 2X^3)T^3 + (2X + 2X^2 + 2X^3)T^4 \\ &+ (2X + 2X^2 + 2X^3 + 2X^4 + X^5)T^5 + (X^2 + X^3 + X^4)T^6 \\ &+ (X + 2X^2 + X^4 + X^5 + X^6)T^7 + (X + 2X^8)T^8, \end{aligned}$$

and therefore an equality in $\mathbb{F}_3[X_0, X_1, T]/(T^9)$

$$\begin{aligned} &E_3(X_0, X_1, 1; T) = \\ &\quad 1 + X_0T + (X_0 + 2X_0^2)T^2 + (X_0 + X_0^2 + X_1)T^3 + (2X_0 + 2X_0^2 + 2X_0^3 + X_0X_1)T^4 \\ &\quad + (2X_0 + 2X_0^2 + 2X_0^3 + 2X_0^4 + X_0^5 + X_0X_1 + 2X_0^2X_1)T^5 \\ &\quad + (X_0^2 + X_0^3 + X_0^4 + X_1 + X_0X_1 + X_0^2X_1 + 2X_1^2)T^6 \\ &\quad + (X_0 + 2X_0^2 + X_0^4 + X_0^5 + X_0^6 + 2X_0^2 + 2X_0^3X_1 + 2X_0X_1^2)T^7 \\ &\quad + (X_0 + 2X_0^8 + X_0^2X_1 + 2X_0^3X_1 + 2X_0^4X_1 + X_0^5X_1 + 2X_0X_1^2 + X_0^2X_1^2)T^8. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \Psi_0(\boldsymbol{X}) &= 1 + 2X_0^3 + X_0^4 + X_0^5 + 2X_0^6 + 2X_0^8 + X_0^2X_1 + 2X_0^3X_1 + 2X_0^4X_1 + X_0^5X_1 + 2X_1^2 + X_0^2X_1^2, \\ \Psi_1(\boldsymbol{X}) &= 2X_0^3 + 2X_0^4 + X_0^6 + 2X_0^8 + X_0X_1 + X_0^2X_1 + X_0^3X_1 + 2X_0^4X_1 + X_0^5X_1 + X_0X_1^2 + X_0^2X_1^2, \\ \Psi_2(\boldsymbol{X}) &= X_0^3 + X_0^4 + 2X_0^5 + 2X_0^8 + 2X_0X_1 + 2X_0^2X_1 + 2X_0^3X_1 + 2X_0^4X_1 + X_0^5X_1 + 2X_0X_1^2 + X_0^2X_1^2, \\ \Psi_3(\boldsymbol{X}) &= 2X_0^4 + 2X_0^6 + 2X_0^8 + 2X_1 + 2X_0^2X_1 + 2X_0^3X_1 + 2X_0^4X_1 + X_0^5X_1 + 2X_1^2 + X_0^2X_1^2, \\ \Psi_4(\boldsymbol{X}) &= X_0^3 + 2X_0^5 + X_0^6 + 2X_0^8 + 2X_0X_1 + 2X_0^2X_1 + X_0^3X_1 + 2X_0^4X_1 + X_0^5X_1 + X_0X_1^2 + X_0^2X_1^2, \\ \Psi_5(\boldsymbol{X}) &= 2X_0^2 + 2X_0^3 + 2X_0^4 + X_0^5 + 2X_0^8 + X_0X_1 + 2X_0^3X_1 + 2X_0^4X_1 + X_0^5X_1 + 2X_0X_1^2 + X_0^2X_1^2, \\ \Psi_6(\boldsymbol{X}) &= 2X_0^2 + X_0^3 + 2X_0^5 + 2X_0^6 + 2X_0^8 + X_1 + 2X_0^3X_1 + 2X_0^4X_1 + X_0^5X_1 + 2X_1^2 + X_0^2X_1^2, \\ \Psi_7(\boldsymbol{X}) &= 2X_0 + 2X_0^2 + X_0^4 + X_0^5 + X_0^6 + 2X_0^8 + X_0^3X_1 + 2X_0^4X_1 + X_0^5X_1 + X_0X_1^2 + X_0^2X_1^2, \\ \Psi_8(\boldsymbol{X}) &= X_0 + 2X_0^8 + X_0^2X_1 + 2X_0^3X_1 + 2X_0^4X_1 + X_0^5X_1 + 2X_0X_1^2 + X_0^2X_1^2. \end{aligned}$$

On the other hand, the endomorphism

$$F - 1 : W_{2,\mathbb{F}_3} = \operatorname{Spec} \mathbb{F}_3[X_0, X_1] \to W_{2,\mathbb{F}_3} = \operatorname{Spec} \mathbb{F}_3[X_0, X_1]$$

is defined by

$$(X_0, X_1) \mapsto (X_0^3 - X_0, X_1^3 - X_1 + X_0^7 - X_0^5).$$

Let R be an $\mathbb{F}_3$-algebra and $a_0, a_1 \in R$. Put

$$S = R[X_0, X_1]/(X_0^3 - X_0 - a_0, X_1^3 - X_1 + X_0^7 - X_0^5 - a_1),$$

and let α_0 and α_1 denote the image of X_0 and X_1 in S, respectively. Then S/R is an unramified cyclic extension of degree 9. The Galois group of S/R is generared by

$$\gamma : (\alpha_0, \alpha_1) \mapsto (\alpha_0 + 1, \alpha_1 - \alpha_0 - \alpha_0^2).$$

Furthermore $\Psi_0(\alpha_0, \alpha_1)$ generates a normal basis of S/R.

Example 4. Let $n = 1$. Substituting

$$U_k = \sum_{j=k}^{p-1} \binom{j}{k} T_j$$

in

$$c_l(\boldsymbol{U}) = c_l\big(\frac{U_1}{U_0}, \frac{U_2}{U_0}, \dots, \frac{U_l}{U_0}\big) \in \mathbb{F}_p\big[\frac{U_1}{U_0}, \frac{U_2}{U_0}, \dots, \frac{U_l}{U_0}\big]$$

defined in 1.5, we obtain rationals

$$\tilde{c}_l(\boldsymbol{T}) = \tilde{c}_l(T_0, T_1, \dots, T_{p-1}) \in \mathbb{F}_p[T_0, T_1, \dots, T_{p-1}, \frac{1}{T_0 + T_1 + \cdots + T_{p-1}}] \ (0 \le l < p).$$

More precisely, we have

$$\tilde{c}_0(\boldsymbol{T}) = \tilde{c}_0(T_0, T_1, \dots, T_{p-1}) = \sum_{j=0}^{p-1} T_j,$$

$$\tilde{c}_1(\boldsymbol{T}) = \tilde{c}_1(T_0, T_1, \dots, T_{p-1}) = \sum_{j=1}^{p-1} jT_j \Big/ \sum_{j=0}^{p-1} T_j$$

and, for $2 \le l < p$, $\tilde{c}_l(\boldsymbol{T}) = \tilde{c}_l(T_0, T_1, \dots, T_{p-1})$ is determined inductively by

$$\sum_{\substack{\nu_1, \nu_2, \dots, \nu_{l-1} \ge 1 \\ \nu_1 + 2\nu_2 + \cdots + (l-1)\nu_{l-1} = l}} \binom{c_1(\boldsymbol{T})}{\nu_1} \binom{c_2(\boldsymbol{T})}{\nu_2} \cdots \binom{c_{l-1}(\boldsymbol{T})}{\nu_{l-1}} + c_l(\boldsymbol{T}) = \sum_{j=l}^{p-1} \binom{j}{l} T_j \Big/ \sum_{j=0}^{p-1} T_j.$$

Then we obtain

$$\mathbb{F}_p[T_0, T_1, \dots, T_{p-1}, \frac{1}{T_0 + T_1 + \cdots + T_{p-1}}] = \mathbb{F}_p[\tilde{c}_0(\boldsymbol{T}), \tilde{c}_1(\boldsymbol{T}), \tilde{c}_2(\boldsymbol{T}), \dots, \tilde{c}_{p-1}(\boldsymbol{T}), \frac{1}{\tilde{c}_0(\boldsymbol{T})}]$$

and

$$\mathbb{F}_p[T_0, T_1, \dots, T_{p-1}, \frac{1}{T_0 + T_1 + \cdots + T_{p-1}}]^{\Gamma} = \mathbb{F}_p[\tilde{c}_0(\boldsymbol{T}), \tilde{c}_1(\boldsymbol{T})^p - \tilde{c}_1(\boldsymbol{T}), \tilde{c}_2(\boldsymbol{T}), \dots, \tilde{c}_{p-1}(\boldsymbol{T}), \frac{1}{\tilde{c}_0(\boldsymbol{T})}].$$

It should be remarked that $\tilde{c}_0(\boldsymbol{T}) = T_0 + T_1 + \cdots + T_{p-1}$ is a group-like element of the Hopf algebra $\mathbb{F}_p[T_0, T_1, \dots, T_{p-1}, 1/(T_0 + T_1 + \cdots + T_{p-1})]$ and $\tilde{c}_1(\boldsymbol{T}), \tilde{c}_2(\boldsymbol{T}), \dots, \tilde{c}_{p-1}(\boldsymbol{T})$ are primitive elements.

Example 4.1. In the case of $p = 2$, we have

$$\tilde{c}_0(\boldsymbol{T}) = T_0 + T_1, \ \tilde{c}_1(\boldsymbol{T}) = \frac{T_1}{T_0 + T_1}$$

and

$$\tilde{c}_1(\boldsymbol{T})^2 - \tilde{c}_1(\boldsymbol{T}) = \frac{T_0 T_1}{T_0 + T_1}.$$

Example 4.2. In the case of $p = 3$, we have

$$\tilde{c}_0(\boldsymbol{T}) = T_0 + T_1 + T_2, \ \tilde{c}_1(\boldsymbol{T}) = \frac{T_1 + 2T_2}{T_0 + T_1 + T_2}, \ \tilde{c}_2(\boldsymbol{T}) = -\frac{T_0T_1 + T_1T_2 + T_2T_0}{(T_0 + T_1 + T_2)^2}$$

and

$$\tilde{c}_1(\boldsymbol{T})^3 - \tilde{c}_1(\boldsymbol{T}) = \frac{(T_0 - T_1)(T_1 - T_2)(T_2 - T_0)}{(T_0 + T_1 + T_2)^3}.$$

Example 4.3. In the case of $p = 5$, we have

$$
\begin{aligned}
\tilde{c}_0(\boldsymbol{T}) &= T_0 + T_1 + T_2 + T_3 + T_4,\\
\tilde{c}_0(\boldsymbol{T})\tilde{c}_1(\boldsymbol{T}) &= T_1 + 2T_2 + 3T_3 + 4T_4,\\
\tilde{c}_0(\boldsymbol{T})^2\tilde{c}_2(\boldsymbol{T}) &= 3(T_0T_1 + T_1T_2 + T_2T_3 + T_3T_4 + T_4T_0) + 2(T_0T_2 + T_1T_3 + T_2T_4 + T_3T_0 + T_4T_1),\\
\tilde{c}_0(\boldsymbol{T})^3\tilde{c}_3(\boldsymbol{T}) &= 3(T_0^2T_1 + T_1^2T_2 + T_2^2T_3 + T_3^2T_4 + T_4^2T_0)\\
&\quad + (T_0T_1^2 + T_1T_2^2 + T_2T_3^2 + T_3T_4^2 + T_4T_0^2) + (T_0^2T_2 + T_1^2T_3 + T_2^2T_4 + T_3^2T_0 + T_4^2T_1)\\
&\quad + 2(T_0T_1T_2 + T_1T_2T_3 + T_2T_3T_4 + T_3T_4T_0 + T_4T_0T_1)\\
&\quad + 3(T_0T_1T_3 + T_1T_2T_4 + T_2T_3T_0 + T_3T_4T_1 + T_4T_0T_2),\\
\tilde{c}_0(\boldsymbol{T})^3\tilde{c}_3(\boldsymbol{T}) &= 4(T_0^3T_1 + T_1^3T_2 + T_2^3T_3 + T_3^3T_4 + T_4^3T_0) + 4(T_0^2T_1^2 + T_1^2T_2^2 + T_2^2T_3^2 + T_3^2T_4^2 + T_4^2T_0^2)\\
&\quad + 2(T_0T_1^3 + T_1T_2^3 + T_2T_3^3 + T_3T_4^3 + T_4T_0^3) + 3(T_0^3T_2 + T_1^3T_3 + T_2^3T_4 + T_3^3T_0 + T_4^3T_1)\\
&\quad + 3(T_0^2T_2^2 + T_1^2T_3^2 + T_2^2T_4^2 + T_3^2T_0^2 + T_4^2T_1^2) + 2(T_0T_2^3 + T_1T_3^3 + T_2T_4^3 + T_3T_0^3 + T_4T_1^3)\\
&\quad + 2(T_0^2T_1T_2 + T_1^2T_2T_3 + T_2^2T_3T_4 + T_3^2T_4T_0 + T_4^2T_0T_1)\\
&\quad + 3(T_0T_1^2T_2 + T_1T_2^2T_3 + T_2T_3^2T_4 + T_3T_4^2T_0 + T_4T_0^2T_1)\\
&\quad + 4(T_0T_1T_2^2 + T_1T_2T_3^2 + T_2T_3T_4^2 + T_4T_0T_1^2 + T_3T_4T_0^2)\\
&\quad + 3(T_0^2T_1T_3 + T_1^2T_2T_4 + T_2^2T_3T_0 + T_3^2T_4T_1 + T_4^2T_0T_2)\\
&\quad + 2(T_0T_1^2T_3 + T_1T_2^2T_4 + T_2T_3^2T_0 + T_3T_4^2T_1 + T_4T_0^2T_2)\\
&\quad + 4(T_0T_1T_3^2 + T_1T_2T_4^2 + T_2T_3T_0^2 + T_3T_4T_1^2 + T_4T_0T_2^2)\\
&\quad + 4(T_0T_1T_2T_3 + T_1T_2T_3T_4 + T_2T_3T_4T_0 + T_3T_4T_0T_2 + T_4T_0T_1T_2)
\end{aligned}
$$

and

$$
\begin{aligned}
&\tilde{c}_0(\boldsymbol{T})^4\big\{\tilde{c}_1(\boldsymbol{T})^5 - \tilde{c}_1(\boldsymbol{T})\big\} =\\
&\quad 4(T_0^4T_1 + T_1^4T_2 + T_2^4T_3 + T_3^4T_4 + T_4^4T_0) + (T_0^3T_1^2 + T_1^3T_2^2 + T_2^3T_3^2 + T_3^3T_4^2 + T_4^3T_0^2)\\
&\quad + 4(T_0^2T_1^3 + T_1^2T_2^3 + T_2^2T_3^3 + T_3^2T_4^3 + T_4^2T_0^3) + (T_0T_1^4 + T_1T_2^4 + T_2T_3^4 + T_3T_4^4 + T_4T_0^4)\\
&\quad + 3(T_0^4T_2 + T_1^4T_3 + T_2^4T_4 + T_3^4T_0 + T_4^4T_1) + 2(T_0^3T_2^2 + T_1^3T_3^2 + T_2^3T_4^2 + T_3^3T_0^2 + T_4^3T_1^2)\\
&\quad + 3(T_0^2T_2^3 + T_1^2T_3^3 + T_2^2T_4^3 + T_3^2T_0^3 + T_4^2T_1^3) + 2(T_0T_2^4 + T_1T_3^4 + T_2T_4^4 + T_3T_0^4 + T_4T_1^4)\\
&\quad + 3(T_0^3T_1T_2 + T_1^3T_2T_3 + T_2^3T_3T_4 + T_3^3T_4T_0 + T_4^3T_0T_1)\\
&\quad + (T_0^2T_1^2T_2 + T_1^2T_2^2T_3 + T_2^2T_3^2T_4 + T_3^2T_4^2T_0 + T_4^2T_0^2T_1)\\
&\quad + 4(T_0T_1^2T_2^2 + T_1T_2^2T_3^2 + T_2T_3^2T_4^2 + T_3T_4^2T_0^2 + T_4T_0^2T_1^2)\\
&\quad + 2(T_0T_1T_2^3 + T_1T_2T_3^3 + T_2T_3T_4^3 + T_3T_4T_0^3 + T_4T_0T_1^3)\\
&\quad + 4(T_0^3T_1T_3 + T_1^3T_2T_4 + T_2^3T_3T_0 + T_3^3T_4T_1 + T_4^3T_0T_2)\\
&\quad + 3(T_0^2T_1T_3^2 + T_1^2T_2T_4^2 + T_2^2T_3T_0^2 + T_3^2T_4T_1^2 + T_4^2T_0T_2^2)\\
&\quad + 2(T_0T_1^2T_3^2 + T_1T_2^2T_4^2 + T_2T_3^2T_0^2 + T_3T_4^2T_1^2 + T_4T_0^2T_2^2)\\
&\quad + (T_0T_1^3T_3 + T_1T_2^3T_4 + T_2T_3^3T_0 + T_3T_4^3T_1 + T_4T_0^3T_2)\\
&\quad + 3(T_0^2T_1T_2T_3 + T_1^2T_2T_3T_4 + T_2^2T_3T_4T_0 + T_3^2T_4T_0T_1 + T_4^2T_0T_1T_2)\\
&\quad + (T_0T_1^2T_2T_3 + T_1T_2^2T_3T_4 + T_2T_3^2T_4T_0 + T_3T_4^2T_0T_1 + T_4T_0^2T_1T_2)\\
&\quad + 4(T_0T_1T_2^2T_3 + T_1T_2T_3^2T_4 + T_2T_3T_4^2T_0 + T_3T_4T_0^2T_1 + T_4T_0T_1^2T_2)\\
&\quad + 2(T_0T_1T_2T_3^2 + T_1T_2T_3T_4^2 + T_2T_3T_4T_0^2 + T_3T_4T_0T_1^2 + T_4T_0T_1T_2^2).
\end{aligned}
$$

Example 5. In the case of $p=2$, $n=2$, the isomorphism

$$\chi^{(1)}\circ\xi : U(\Gamma)_{\mathbb{F}_2} = \operatorname{Spec}\mathbb{F}_2[T_0,T_1,T_2,T_3,\frac{1}{T_0+T_1+T_2+T_3}]$$
$$\xrightarrow{\sim} \mathbb{G}_{m,\mathbb{F}_2}\times W_{2,\mathbb{F}_2}\times \mathbb{G}_{a,\mathbb{F}_2} = \operatorname{Spec}\mathbb{F}_2[X_0,X_1,X_2,X_3,\frac{1}{X_0}]$$

is defined by

$$X_0\mapsto \tilde{c}_0(\boldsymbol{T}),\ X_1\mapsto \tilde{c}_1(\boldsymbol{T}),\ X_2\mapsto \tilde{c}_2(\boldsymbol{T}),\ X_3\mapsto \tilde{c}_3(\boldsymbol{T}),$$

where

$$T_0+T_1(1+U)+T_2(1+U)^2+T_3(1+U)^3 =$$
$$\tilde{c}_0(\boldsymbol{T})\big\{1+\tilde{c}_1(\boldsymbol{T})U+(\tilde{c}_1(\boldsymbol{T})+\tilde{c}_1(\boldsymbol{T})^2+\tilde{c}_2(\boldsymbol{T}))U^2+(\tilde{c}(\boldsymbol{T})+\tilde{c}_1(\boldsymbol{T})^2+\tilde{c}_1(\boldsymbol{T})\tilde{c}_2(\boldsymbol{T}))U^3\big\}\big\{1+\tilde{c}_3(\boldsymbol{T})U\big\}$$

in $\mathbb{F}_2[T_0,T_1,T_2,T_3][U]/(U^4)$. Hence we obtain

$$\begin{aligned}
\tilde{c}_0(\boldsymbol{T}) &= T_0+T_1+T_2+T_3,\\
\tilde{c}_0(\boldsymbol{T})\tilde{c}_1(\boldsymbol{T}) &= T_1+T_3,\\
\tilde{c}_0(\boldsymbol{T})^2\tilde{c}_2(\boldsymbol{T}) &= T_2^2+T_3^2+T_0T_1+T_0T_2+T_1T_3+T_2T_3,\\
\tilde{c}_0(\boldsymbol{T})^3\tilde{c}_3(\boldsymbol{T}) &= (T_0^2T_1+T_1^2T_2+T_2^2T_3+T_3^2T_0)+(T_0T_1T_2+T_1T_2T_3+T_2T_3T_0+T_3T_0T_1).
\end{aligned}$$

Furthermore we have

$$\mathbb{F}_2[T_0,T_1,T_2,T_3,\frac{1}{T_0+T_1+T_2+T_3}]^{\Gamma} =$$
$$\mathbb{F}_2[\tilde{c}_0(\boldsymbol{T}),\tilde{c}_1(\boldsymbol{T})^2+\tilde{c}_1(\boldsymbol{T}),\tilde{c}_2(\boldsymbol{T})^2+\tilde{c}_2(\boldsymbol{T})+\tilde{c}_1(\boldsymbol{T})^3+\tilde{c}_1(\boldsymbol{T})^2,\tilde{c}_3(\boldsymbol{T}),\frac{1}{\tilde{c}_0(\boldsymbol{T})}].$$

We have gotten

$$\begin{aligned}
&\tilde{c}_0(\boldsymbol{T})^2\big\{\tilde{c}_1(\boldsymbol{T})^2+\tilde{c}_1(\boldsymbol{T})\big\} =\\
&\qquad T_0T_1+T_1T_2+T_2T_3+T_3T_0,\\
&\tilde{c}_0(\boldsymbol{T})^4\big\{\tilde{c}_2(\boldsymbol{T})^2+\tilde{c}_2(\boldsymbol{T})+\tilde{c}_1(\boldsymbol{T})^3+\tilde{c}_1(\boldsymbol{T})^2\big\} =\\
&\qquad (T_0^3T_1+T_1^3T_2+T_2^3T_3+T_3^3T_0)+(T_0^3T_2+T_1^3T_3+T_2^3T_0+T_3^3T_1)+\\
&\qquad (T_0T_1^2T_2+T_1T_2^2T_3+T_2T_3^2T_0+T_3T_0^2T_1)+(T_0T_1T_2^2+T_1T_2T_3^2+T_2T_3T_0^2+T_3T_0T_1^2)
\end{aligned}$$

by a simple and honest calculation.