Notation

- For any ring A, $\zeta(A)$ denotes the center of A,
- K/k : a finite Galois extension of number fields with Galois group G, and we assume K is a CM-field and k is a totally real field),
- S: a finite set of places of k which contains $S_\infty \cup S_{ram}$,
- T: another finite set of places of k such that
  - $\underline{T \cap S = \emptyset}$,
  - $\underline{\beta \not\equiv 1 \mod \prod_{\mathfrak{P} \in T(K)} \mathfrak{P} \text{ for all non-trivial roots of unity } \beta \text{ in } K}$.

We will refer the above underlined conditions to $Hyp(S,T)$.
The Stickelberger element for $(S, T, K/k)$

$$\theta_{S,K/k}^T := \sum_{\chi \in \mathrm{Irr}\, G, \chi:\mathrm{odd}} \prod_{\mathfrak{p} \in T} \det(1 - \phi_{\mathfrak{P}}^{-1} N\mathfrak{p} \mid V_\chi) L_S(0, \bar{\chi}, K/k) e_\chi \in \zeta(\mathbb{Q}[G])$$

where

- $\mathfrak{P}$ is a fixed prime of K above $\mathfrak{p}$,
- $\phi_{\mathfrak{P}}$ is the Frobenius automorphism at $\mathfrak{P}$,
- V_χ is a irreducible representation of G with χ,
- $\bar{\chi}$ is the contragredient character of χ,
- $e_\chi = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma^{-1})\sigma \in \zeta(\mathbb{C}[G])$.

We need more notation.

- $\mathfrak{m}(G)_p$: a maximal $\mathbb{Z}_p$-order in $\mathbb{Q}_p[G]$ which contains $\mathbb{Z}_p[G]$,
- $\mathrm{nr} : \mathbb{Q}_p[G] \to \zeta(\mathbb{Q}_p[G])$: the reduced norm of $\mathbb{Q}_p[G]$,
  - If G is abelian, nr is just the identity map.
  - $\mathrm{nr}(\mathbb{Z}_p[G])$ is not contained in $\zeta(\mathbb{Z}_p[G])$ in general but in $\zeta(\mathfrak{m}(G)_p)$,
  - We extend nr to any matrix ring over $\mathbb{Q}_p[G]$.
- $\mathcal{I}(G)_p := \langle \mathrm{nr}(H) \mid H \in M_n(\mathbb{Z}_p[G]),\ \forall n \in \mathbb{N} \rangle_{\zeta(\mathbb{Z}_p[G])} \subset \zeta(\mathfrak{m}(G)_p)$,
- $\mathcal{H}(G)_p := \{x \in \zeta(\mathbb{Z}_p[G]) \mid xH^* \in M_n(\mathbb{Z}_p[G]),\ \forall H \in M_n(\mathbb{Z}_p[G],\ \forall n \in \mathbb{N}\}$,
  - H^* is a matrix in $M_n(\mathfrak{m}(G)_p)$ such that $HH^* = H^*H = \mathrm{nr}(H)1_{n \times n}$.
  - If G is abelian, H^* is the adjoint matrix of H.
- $\mathfrak{F}(G)_p := \{x \in \zeta(\mathbb{Z}_p[G]) \mid x\mathfrak{m}(G)_p \subset \mathbb{Z}_p[G]\}$:the central conductor of $\mathfrak{m}(G)_p$ over $\mathbb{Z}_p[G]$,
- $\mathfrak{A} := \langle \mathrm{nr}(\prod_{\mathfrak{p} \in T}(1 - \phi_{\mathfrak{P}}^{-1} N\mathfrak{p})) \mid Hyp(S,T) \text{ is satisfied} \rangle_{\zeta(\mathbb{Z}_p[G])}$,
  - If G is abelian, $\mathfrak{A} = \mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K))$.
  - We set $\delta_T := \mathrm{nr}(\prod_{\mathfrak{p} \in T}(1 - \phi_{\mathfrak{P}}^{-1} N\mathfrak{p}))$.
- $\omega_{K,p} := \mathrm{nr}(\sharp\mu(K)_p)$,
- For any $\alpha \in K$, $S_\alpha := \{\mathfrak{p} \mid \ \mathfrak{p} \text{ is a prime of } k \text{ such that } \mathfrak{p} | N_{K/k}(\alpha)\}$.

Formulation of the conjectures

weak Non-abelian Brumer's conjecture $(B_w(K/k,S)_p)$**.**

$1_{B_{w,p}}$ $\mathfrak{A}\theta_{S,K/k} \subset \mathcal{I}(G)_p(\zeta(\mathfrak{m}(G)_p))$.

$2_{B_{w,p}}$ *For any* $x \in \mathcal{H}(G)_p(\mathfrak{F}(G)_p)$, $x\mathfrak{A}\theta_{S,K/k} \subset \mathrm{Ann}_{\mathbb{Z}_p[G]}(Cl(K)\otimes\mathbb{Z}_p)$.

weak Non-abelian Brumer-Stark conjecture $(BS_w(K/k,S)_p)$**.**

$1_{BS_{w,p}}$ *For any fractional ideal* $\mathfrak{D}$ *of* K *and for any* $x \in \mathcal{H}(G)_p(\mathfrak{F}(G)_p)$, *there exists anti-unit* $\alpha = \alpha(\mathfrak{D}, x, S) \in K$ *such that*

$$\mathfrak{D}^{x\omega_K\theta_{S,K/k}} = (\alpha)$$

$2_{BS_{w,p}}$ *and for any* T *which satisfies* $Hyp(S \cup S_\alpha, T)$, *there exists* $\alpha_T \in \mathfrak{o}^*_{S_\alpha,K}$ *such that*

$$\alpha^{z\delta_T} = \alpha_T^{x\omega_{K,p}}$$

for any $z \in \mathcal{H}(G)_p(\mathfrak{F}(G)_p)$.

Observation;

1. If G is abelian, $\begin{cases} 1_{B_p} \Leftrightarrow \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mu(K)_p)\theta_{S,K/k} \subset \mathbb{Z}_p[G], \\ 1_{B_{w,p}} \Leftrightarrow \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mu(K)_p)\theta_{S,K/k} \subset \mathfrak{m}(G)_p, \\ 2_{B_p} \Leftrightarrow \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mu(K)_p)\theta_{S,K/k} \subset \mathrm{Ann}_{\mathbb{Z}_p[G]}(Cl(K)\otimes\mathbb{Z}_p), \\ 2_{B_{w,p}} \quad |G|\,\mathrm{Ann}_{\mathbb{Z}_p[G]}(\mu(K)_p)\theta_{S,K/k} \subset \mathrm{Ann}_{\mathbb{Z}_p[G]}(Cl(K)\otimes\mathbb{Z}_p), \\ 2_{BS_p} \Leftrightarrow K(\alpha^{1/\sharp\mu(K)_p})/k : \text{abelian extension} \\ 2_{BS_{w,p}} \Leftrightarrow K(\alpha^{|G|/\sharp\mu(K)_p})/k : \text{abelian extension} \end{cases}$

2. If $\mathcal{I}(\mathbb{Z}_l[G]) = \zeta(\mathfrak{m}(G)_l)$, $B_{w,l} \Leftrightarrow B_l$ and $BS_{w,l} \Leftrightarrow BS_l$.

- $l \nmid |G| \Rightarrow \mathcal{I}(\mathbb{Z}_l[G]) = \zeta(\mathfrak{m}(G)_l)$
- l is odd and $G \cong D_{4l} \Rightarrow \mathcal{I}(\mathbb{Z}_l[G]) = \zeta(\mathfrak{m}(G)_l)$

Known Results

Let p be any odd prime.

1. (Nickel, 2011)

If p is "non-exceptional" and $\mu = 0$, $B(K/k,S)_p$ and $BS(K/k.S)_p$ are true.

"non-exceptional" $\Leftrightarrow$ $\begin{cases} p \text{ is odd}, \\ \text{all primes } \mathfrak{p} \text{ above } p \text{ are tamely ramified or } j \in G_{\mathfrak{P}}, \\ K^{cl} \not\subset (K^+)^{cl}(\zeta_p). \end{cases}$

2. (Nickel, preprint)

If $S \supset S_p$ and $\mu = 0$, $B(K/k,S)_p$ and $BS(K/k.S)_p$ are true.

3. (Burns, preprint)

"The Gross-Stark conjecture for p" + ETNC $+\mu = 0$ implies $B(K/k,S)_p$ and $BS(K/k.S)_p$.

For the weak conjecture,

4. (Burns-Johnston, 2011)

If p is unramified in $K/\mathbb{Q}$ and every Inertia subgroup is normal in G, $B_w(K/k,S)_p$ is true.

5. (Nickel, 2011)

If no prime above p splits in K/K^+ whenever $K^{cl} \subset (K^{cl})^+(\zeta_p)$, $B_w(K/k,S)_p$ and $BS_w(K/k,S)$ are true.

Main Results

Assume G is supersolvable and $S \supset S_\infty$(S does not hove to contain S_{ram})

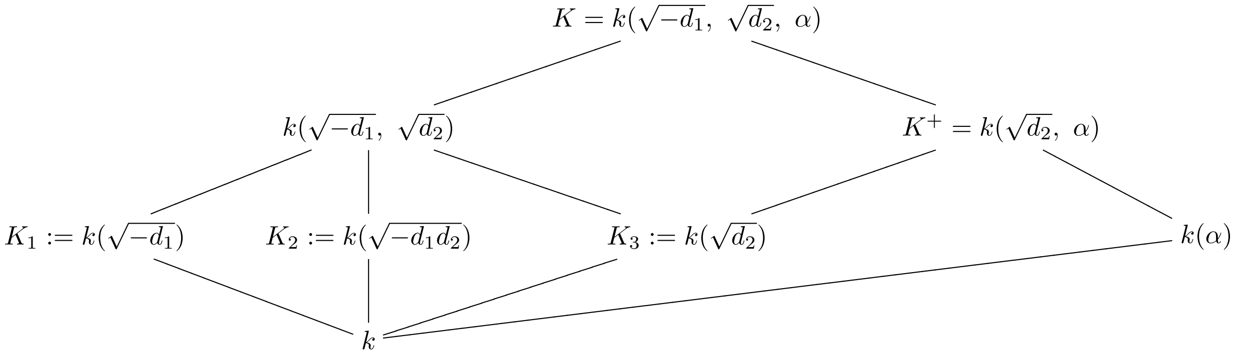
Proposition 1. $\mathfrak{A}_{S \cup S}\theta_{K/k,S} \subset \zeta(\mathfrak{m}(G))$

Theorem 1. *If $B_w(K'/k')$(resp. $BS_w(K'/k')$) is true for certain subabelian extensions K'/k' in K/k, $B_w(K/k,S)$(resp. $BS_w(K/k,S)$) is true.*

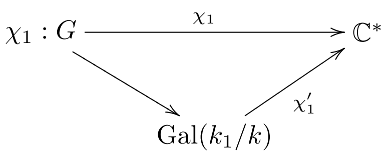
Theorem 2. *If $G \cong D_{4p}$ or $Q_{2^{n+2}}$, $B_w(K/k,S)$ and $BS_w(K/k,S)$are true.*

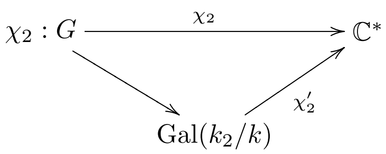
Settings for the proof in the case $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_3 (= D_{12})$

We use the presentation $\mathfrak{S}_3 = \langle \sigma,\ \tau \mid \sigma^3 = \tau^2 = 1,\ \tau\sigma\tau = \sigma^2 \rangle$.



where d_1 and d_2 are totally positive elements in k, α is a root of some cubic equation, $\mathrm{Gal}(k_1/k)$ corresponds to $\mathbb{Z}/2\mathbb{Z}$, $\mathrm{Gal}(K^+/k)$ corresponds to $\mathfrak{S}_3$ and K/k_3 is a cyclic extension of degree 6 whose Galois group is $\langle \sigma j \rangle$. G has three odd characters $\chi_1,\ \chi_2$ and χ_3 as follows:





where χ_1' and χ_2' are non-trivial characters of $\mathrm{Gal}(K1/k)$ and $\mathrm{Gal}(K_2/k)$ respectively.
Let ϕ be the irreducible character of $\mathrm{Gal}(K/K_3)$ which sends σ and j to ζ_3 and -1 respectively. Then we have

$$\chi_3 = \mathrm{Ind}^G_{\mathrm{Gal}(K/k_3)}(\phi).$$