On the Fourier coefficients of Jacobi forms of index N over totally real number fields

No.

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§1 Jacobi forms over totally real number fields

総実代数体上のJacobi形式のFourier係数とHilbertモジュラ形式のzeta関数の中心値との間の関係を与える。

F: a totally real number field of degree n with class number one

$\mathcal{O}$: the maximal order of F

d_F: the discriminant of F, δ: the different of F relative to $\mathbb{Q}$

E: the unit group of F

$\tau_1, \dots, \tau_n$: the isomorphisms of F to $\mathbb{R}$.

For $\alpha \in F$, put $\alpha^{(\nu)} = \tau_\nu(\alpha)$ $(1 \leqq \nu \leqq n)$.

For $N \in \mathcal{O}$ $(N \gg 0)$, we put

(1-1) $\tilde{\Gamma}_0(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}) \mid N | C \text{ and } \det \gamma \gg 0 \right\}$

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma}_0(N)$ acts on H^n and $H^n \times \mathbb{C}^n$ by

(1-2) $\tau \longrightarrow \gamma(\tau) = \left(\dfrac{a^{(1)}\tau_1 + b^{(1)}}{c^{(1)}\tau_1 + d^{(1)}}, \dots, \dfrac{a^{(n)}\tau_n + b^{(n)}}{c^{(n)}\tau_n + d^{(n)}} \right)$ $\quad (\tau = (\tau_1, \dots, \tau_n) \in H^n)$

$(\tau, z) \longrightarrow \gamma(\tau, z) = \left(\gamma(\tau); \dfrac{z_1}{c^{(1)}\tau_1 + d^{(1)}}, \dots, \dfrac{z_n}{c^{(n)}\tau_n + d^{(n)}} \right)$ $\quad \left(\begin{matrix} \tau = (\tau_1, \dots, \tau_n) \in H^n \\ z = (z_1, \dots, z_n) \in \mathbb{C}^n \end{matrix} \right)$

$(\lambda, \mu) \in \mathcal{O}^2$ acts on $H^n \times \mathbb{C}^n$ by

(1-3) $(\tau, z) \longrightarrow (\lambda, \mu)(\tau, z) = (\tau_1, \dots, \tau_n; z_1 + \lambda^{(1)}\tau_1 + \mu^{(1)}, \dots, z_n + \lambda^{(n)}\tau_n + \mu^{(n)})$

a holomorphic function $\phi(\tau, z)$ on $H^n \times \mathbb{C}^n$ satisfying

(i) $\phi(\gamma(\tau, z)) = (c\tau + d)^k e\left[tr\left(\frac{N}{\delta}\left(\frac{cz^2}{c\tau + d}\right)\right)\right] \phi(\tau, z)$ $\quad \left(\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}) \right)$

(1-4) (ii) $\phi((\lambda, \mu)(\tau, z)) = e\left[-tr\left(\frac{N}{\delta}(\lambda^2 + 2\lambda z)\right)\right] \phi(\tau, z)$ $\quad (\forall (\lambda, \mu) \in \mathcal{O}^2)$

(iii) $\phi(\tau, z) = \sum_{\substack{(n, r) \in \mathcal{O}^2 \\ 4Nn - r^2 \gg 0}} c(n, r)\, e\left[tr\left(\frac{n}{\delta}\tau + \frac{r}{\delta}z\right)\right]$,

where $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$. $J_{k,N}^{cusp} = \{\phi\}$. We call ϕ a Jacobi form of index N and weight k over F

For $\rho, \Delta \in \mathcal{O}$ such that $\rho \in \mathcal{O}/2N\mathcal{O}$, $\Delta \gg 0$ and $\Delta \equiv \rho^2 \pmod{4N}$, we put

(1-5) $L_{N,\Delta,\rho} = \left\{ Q = [Na, b, c] = \begin{pmatrix} Na & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \,\middle|\, \begin{array}{l} a, b, c \in \mathcal{O},\ b^2 - 4Nac = \Delta \\ \text{and } b \equiv \rho \pmod{2N} \end{array} \right\}$

We impose the following condition:

$D_0 \in \mathcal{O}$, $D_0 \mid \Delta$, Δ/D_0 is square in $\mathcal{O}/4N\mathcal{O}$, $D_0 \ll 0$, $(D_0, 4N) = 1$

(1-6) the finite part of the conductor of the quadratic extension $F(\sqrt{D_0})/F$ equals (D_0) and

$$D_0 = \pi_1^* \cdots \pi_l^*$$

注4に関して

with distinct primary odd prime elements π_i^* of F $(1 \leqq i \leqq l)$.
ここで integer π_i^* ~~is odd~~ がFのintegerの平方と合同のとき primary という。

For $\alpha, \beta \in \mathcal{O}$ satisfying $(2, \beta) = 1$, we define $\left(\frac{\alpha}{\beta}\right)$

$\left(\frac{\alpha}{\beta}\right) = \prod_{i=1}^{s} \left(\frac{\alpha}{\mathfrak{p}_i}\right)^{e_i}$, $\left(\frac{\alpha}{\mathfrak{p}_i}\right) = \#\{x \in \mathcal{O}/\mathfrak{p}_i \mid x^2 \equiv \alpha \pmod{\mathfrak{p}_i}\} - 1$, where

$(\beta) = \prod_{i=1}^{s} \mathfrak{p}_i^{e_i}$, $\mathfrak{p}_i$ $(1 \leqq i \leqq s)$ is prime.

We define a genus character $\chi_{D_0}(Q)$ by

(1-7) $\chi_{D_0}(Q) = \begin{cases} \left(\frac{m}{D_0}\right) & \text{if } (a, b, c, D_0) = 1, \\ 0 & \text{otherwise} \end{cases}$

Heckeの平方剰余記号

for every $Q = [Na, b, c] \in L_{N,\Delta,\rho}$, where m is an element of $\mathcal{O}$ such that $(m, D_0) = 1$, $m = aN_1x^2 + bxy + cN_2y^2$ for some N_1, N_2, x and $y \in \mathcal{O}$ with $N = N_1N_2$ and $N_1 \gg 0$, $N_2 \gg 0$.

§2. A correspondence from Jacobi forms to Hilbert modular forms. Let $k=(k_1,\dots,k_n)\in\mathbb{Z}^n$ $(k_i>1)$. We put

$$S_{2k}(\tilde{\Gamma}_0(N))=\left\{\text{cusp forms } f \text{ of weight } 2k \text{ satisfying } f(\gamma(z))=\det\gamma^{-k}(cz+d)^{2k}f(z) \text{ for every } \gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\tilde{\Gamma}_0(N)\right\}$$

Define

$$\underline{(2\text{-}1)}\quad f_k(z)=f_{k,N,\Delta,\rho,D_0}(z)=\sum_{Q\in L_{N,\Delta,\rho}}\frac{\chi_{D_0}(Q)}{Q(z,1)^k}\quad\text{for every } z=(z_1,\dots,z_n)\in H^n$$

where $Q(z,1)^k=(aNz^2+bz+c)^k$ $\left(Q=\begin{pmatrix}Na&\frac{b}{2}\\\frac{b}{2}&c\end{pmatrix}\right)$. Then $f_k\in M_{2k}(N)^{\operatorname{sgn}D_0}$

$$=\left\{f\in S_{2k}(\tilde{\Gamma}_0(N))\ \middle|\ f\left(-\frac{1}{Nz}\right)=(-Nz^2)^k\prod_{i=1}^n\operatorname{sgn}D_0^{(i)}f(z)\right\}.$$

Let $(n,r)\in\mathcal{O}^2$ $(r^2-4Nn\ll 0)$, define

重さ $k+1$

$$\underline{(2\text{-}2)}\quad P_k=P_{k,N,(n,r)}(\tau,z)=\sum_{\gamma\in\Gamma_\infty^J(1)\backslash\Gamma^J(1)}e^{n,r}|_{k,N}\gamma(\tau,z),$$

は index N の Jacobi ~~forms~~ の Poincaré 級

where $e^{n,r}(\tau,z)=e[\operatorname{tr}(\frac{n}{\delta}\tau+\frac{r}{\delta}z)]$, $\Gamma^J(1)=\{(\gamma,(\lambda,\mu))\mid\gamma\in SL_2(\mathcal{O}),\lambda,\mu\in\mathcal{O}\}$
$\Gamma_\infty^J(1)=\{(\begin{pmatrix}1&n\\0&1\end{pmatrix},(0,\mu))\mid n,\mu\in\mathcal{O}\}$.

Here, for a function ϕ on $H^n\times\mathbb{C}^n$ and $(\gamma,(\lambda,\mu))\in\Gamma^J(1)$, we put

$$\phi|_{k,N}(\gamma,(\lambda,\mu))(\tau,z)=(c\tau+d)^{-k}\,e\left[\operatorname{tr}\left(\frac{N}{\delta}\left(\frac{-c(z+\lambda\tau+\mu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z+\lambda\mu\right)\right)\right]\phi(\gamma,(\lambda,\mu))(\tau,z).$$

Define an function $\Omega_k=\Omega_{k,N,D_0,r_0}(w;\tau,z)$ on $H^n\times(H^n\times\mathbb{C}^n)$

$$\Theta_{k,N;D_0,\gamma_0}(w;\tau,z) = c_{k,N,D_0}$$

$$\underline{(2-3)} \quad \times \sum_{\substack{(n,\gamma)\in\mathcal{O}^2 \\ 4Nn-\gamma^2 \gg 0}} (4Nn-\gamma^2)^{k-\frac{1}{2}} f_{k,N,D_0(\gamma^2-4Nn),\gamma_0\gamma,-D_0}(w)\, e[tr(\tfrac{n}{\delta}\tau + \tfrac{\gamma}{\delta}z)]$$

By computing Fourier coefficients of both hand, we deduce the following theorem.

Theorem 2-1. Suppose that $D_0 = \gamma_0^2 - 4Nn_0$ satisfies the assumption (1.6). Then

$$\Theta_{k,N,D_0,\gamma_0}(w;\tau,z) = \tau_{k,N,D_0}$$

$$\underline{(2-4)} \quad \times \sum_{\substack{m\in\mathcal{O} \\ m\gg 0}} m^{k-1} \Big(\sum_{\substack{dd'=m \\ d\in\mathcal{O}^+/E^+}} \Big(\frac{d}{D_0}\Big)(d')^k P_{k+1,N,n_0(d')^2,\gamma_0 d'}(\tau,z)\, e[tr(\frac{mw}{\delta})] \Big).$$

Theorem 2-1 は次の基本 ~~Proposition~~ 2.2 に帰着できる。

Let $\gamma_0, n_0, \gamma, n'$ and $b\in\mathcal{O}$ denote elements such that

$D_0 = \gamma_0^2 - 4Nn_0$, $D = \Delta/D_0 = \gamma^2 - 4Nn'$ and $b \equiv \gamma_0\gamma \pmod{2N}$

Given an integral ideal (a) in F, we define a sum F_a by

$$F_a = F_a(N,\gamma_0,n_0,\gamma,s,n') = N(a)^{-1} \sum_{\lambda\in(\mathcal{O}/a\mathcal{O})^\times} \sum_{x,y\in\mathcal{O}/a\mathcal{O}} e[tr(\frac{\lambda F(x,y)}{a\delta})]$$

with $F(x,y) = Nx^2 + \gamma_0 xy + n_0 y^2 + \gamma x + sy + n'$ with $s = \dfrac{\gamma_0\gamma - b}{2N}$.

We deduce the following proposition.

Proposition 2.2. Suppose that D_0 satisfies the above condition. Then

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$$N(a)^{-1}\sum_{(d)|a,\, d\gg 0}\left(\frac{d}{D_0}\right)F_{\frac{a}{d}}=\begin{cases}\chi_{D_0}([Na, b, \frac{b^2-\Delta}{4Na}]) & \text{if } a \mid \frac{b^2-\Delta}{4N}\\ 0 & \text{otherwise}\end{cases}$$

proof. we may reduce the case where $a=\pi^l$ with a positive prime element π. We assume that $(\pi, 2)=1$ and $\pi \nmid D_0$.
We have

$$F_a=\sum_{\lambda(a)^*}\left(\frac{D_0}{a}\right)e[\mathrm{tr}(\lambda C/\delta a)] \text{ with } C=Ns^2+\gamma_0(-s\gamma)+n_0\gamma^2+nD$$

$$\text{Therefore }\sum_{(d)|a,\, d\gg 0}\left(\frac{d}{D_0}\right)F_{\frac{a}{d}}=\sum_{(d)|a,\, d\gg 0}\left(\frac{d}{D_0}\right)\left(\frac{D_0}{\frac{a}{d}}\right)\sum_{\substack{\lambda(a)^*\\(\lambda,a)=d}}e[\mathrm{tr}(\lambda C/\delta a)]$$

Applying the quadratic reciprocity law, we find that

$$\sum_{(d)|a,\, d\gg 0}\left(\frac{d}{D_0}\right)F_{\frac{a}{d}}=\left(\frac{a}{D_0}\right)\sum_{\lambda(a)}e[\mathrm{tr}(\lambda C/\delta a)]=\begin{cases}\left(\frac{a}{D_0}\right)N(a) & \text{if } a \mid C,\\ 0 & \text{otherwise}\end{cases}$$

Define a function $\Psi_{D_0,\gamma_0}(\phi)$ on H^n by

(9-5) $\Psi_{D_0,\gamma_0}(\phi)(w)=\langle \phi, \Omega_{k,N,D_0,\gamma_0}(-\bar{w};*)\rangle \quad \forall\, \phi\in J^{cusp}_{k+1,N}$. Then $\Psi_{D_0,\gamma_0}(\phi)\in M_{2k}(N)^{\mathrm{sgn} D_0}$ and Fourier coefficients of it are determined by

$$\Psi_{D_0,\gamma_0}(\phi)(w)=\sum_{\substack{m\in\mathcal{O}\\ m\gg 0}}\left(\sum_{\substack{(d)|m\\ d\gg 0}}\left(\frac{d}{D_0}\right)d^{k-1}c\left(\left(\frac{m}{d}\right)^2 n_0, \frac{m}{d}\gamma_0\right)\right)e\left[\mathrm{tr}\left(\frac{mw}{\delta}\right)\right]$$
(9-6)

for every $\phi(\tau,z)=\sum_{n,\gamma}c(n,\gamma)e[\mathrm{tr}(\frac{n}{\delta}\tau+\frac{\gamma}{\delta}z)]\in J^{cusp}_{k+1,N}$

次の図は可換である。上下の矢印はHecke operatorsを表わす。

$$\begin{array}{ccc} J^{cusp}_{k+1,N} & \xrightarrow{\Psi_{D_0,r_0}} & M_{2k}(N)^{\mathrm{sgn} D_0} \\ \downarrow T_{k+1,N}(\wp) & & \downarrow T_{2k,N}(\wp) \\ J^{cusp}_{k+1,N} & \xrightarrow{\Psi_{D_0,r_0}} & M_{2k}(N)^{\mathrm{sgn} D_0} \end{array}$$

§ 3. Fourier coefficients of Jacobi forms.

We deduce the following Theorem

Theorem 3.1. Suppose that $D_0 = r_0^2 - 4Nn_0$ satisfies the assumption (1-6) and ϕ is an eigenfunction of Hecke operators on $J^{cusp}_{k+1,N}$, $\Psi_{D_0,r_0}(\phi)$ is new form in $M_{2k}(N)^{\mathrm{sgn} D_0}$ and ϕ satisfies the condition about multiplicity one theorem. Then

$$(3\text{-}1)\quad |C(n_0,r_0)|^2 \frac{\langle f,f\rangle}{\langle \phi,\phi\rangle} = (k-1)!\,\frac{\delta^{k-\frac{3}{2}}|D_0|^{k-\frac{1}{2}}}{2^{2k-1}N^{k-1}\pi^k}(E^+:E_0)\,D\!\left(k_0,\chi,\left(\frac{*}{D_0}\right)\right),$$

where f is the primitive form associated with $\Psi_{D_0,r_0}(\phi)$, χ is the eigenvalue of ~~with respect to~~ f with respect to Hecke operators,

$$D\!\left(s,\chi,\left(\frac{*}{D_0}\right)\right) = \prod_{\substack{\wp | 2,\ \wp=(\pi),\ \pi\gg 0}} \left(1-\chi(\wp)\left(\frac{\pi}{D_0}\right)N(\wp)^{-s} + 1_N(\wp)\left(\frac{\pi}{D_0}\right)^2 N(\wp)^{2k_0-1-2s}\right)^{-1}$$

$$\times \prod_{\wp\nmid 2}\left(1-\chi(\wp)\left(\frac{F(\sqrt{D_0})/F}{\wp}\right)N(\wp)^{-s} + 1_{N(\wp)}\left(\frac{F(\sqrt{D_0})/F}{\wp}\right)^2 N(\wp)^{2k_0-1-2s}\right)^{-1},$$

$$1_N(\wp) = \begin{cases} 1 & \text{if } \wp\nmid N \\ 0 & \text{otherwise} \end{cases},\quad \left(\frac{F(\sqrt{D_0})/F}{\wp}\right) \text{ is the Artin symbol}$$

$$E^+ = \{\alpha\in E \mid \alpha^{(i)}>0\ (1\le i\le n)\},\quad E_0 = \{\varepsilon^2 \mid \varepsilon\in E,\ \varepsilon^2\equiv 1 \pmod{D_0}\}$$

$$\langle f,g\rangle = \mu(\tilde{\Gamma}_0(N)\backslash H^n)^{-1}\int_{\tilde{\Gamma}_0(N)\backslash H^n} \overline{f(z)}\,g(z)\,y^{2k}\,\frac{dx\,dy}{y^2}$$

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