The zeta-function of the root system of type G_2

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(a joint work with Y.Komori and H.Tsumura)

V : r-dimensional real vector space

$\langle\ ,\ \rangle$: inner product defined on V

Δ : finite reduced root system in V

$\Delta = \Delta_+ \cup \Delta_-$

$\Psi = \{\alpha_1, \ldots, \alpha_r\}$: fundamental system

$\{\lambda_1, \ldots, \lambda_r\}$: fundamental weights (with $\langle \alpha_i^\vee, \lambda_j \rangle = \delta_{ij}$)

$P_{++} = \oplus_{i=1}^r \mathbb{N}\lambda_i \qquad (\mathbb{N} = \mathbb{Z}_{\geq 1})$

Define the zeta-function of Δ by

$$\zeta_r(\mathbf{s}, \Delta) = \sum_{\lambda \in P_{++}} \prod_{\alpha \in \Delta_+} \frac{1}{\langle \alpha^\vee, \lambda \rangle^{s_\alpha}}$$

where $\mathbf{s} = (s_\alpha)_{\alpha \in \Delta_+} \in \mathbb{C}^n$, $n = |\Delta_+|$.

Examples.

Ex 1. $\Delta = A_1$

$$\Rightarrow \Delta_+ = \{\alpha_1\},\ P_{++} = \{m\lambda_1 \mid m \in \mathbb{N}\},$$

$$\Rightarrow \zeta_1(s, A_1) = \textstyle\sum_{\lambda\in P_{++}} \langle \alpha_1^\vee, \lambda\rangle^{-s} = \sum_{m=1}^\infty m^{-s} = \zeta(s)$$

(the Riemann zeta-function)

Ex 2. $\Delta = A_2$

$$\Rightarrow \Delta_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\},$$

$$P_{++} = \{m_1\lambda_1 + m_2\lambda_2 \mid m_1, m_2 \in \mathbb{N}\},$$

Therefore

$$\zeta_2(s, A_2) = \sum_{\lambda\in P_{++}} \prod_{\alpha\in\Delta_+} \frac{1}{\langle \alpha^\vee, \lambda\rangle^{s_\alpha}} = \sum_{m_1=1}^\infty \sum_{m_2=1}^\infty m_1^{-s_1} m_2^{-s_2} (m_1+m_2)^{-s_3}$$

(the Tornheim double sum)

Ex 3. The case $\mathbf{s} = (s, s, \ldots, s)$

$\mathfrak{g}$: a semisimple Lie algebra over $\mathbb{C}$

$\Delta = \Delta(\mathfrak{g})$

Witten zeta-function : $\zeta_W(s, \mathfrak{g}) = \sum_{\varphi} (\dim \varphi)^{-s}$

(φ runs over all finite dim irreducible representation of $\mathfrak{g}$)

By Weyl's dimension formula we have

$$\begin{aligned}
&\zeta_W(s, \mathfrak{g}) \\
&= \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{\alpha \in \Delta_+} \left(\frac{\langle \alpha^\vee, m_1 \lambda_1 + \cdots + m_r \lambda_r \rangle}{\langle \alpha^\vee, \lambda_1 + \cdots + \lambda_r \rangle} \right)^{-s} \\
&= \left(\prod_{\alpha \in \Delta_+} \langle \alpha^\vee, \lambda_1 + \cdots + \lambda_r \rangle \right)^{s} \zeta_r((s, s, \ldots, s), \Delta(\mathfrak{g}))
\end{aligned}$$

Ex 4. The Euler-Hoffman-Zagier multiple sum

$$\zeta_{EHZ,r}(\mathbf{s}) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} n_1^{-s_1}(n_1+n_2)^{-s_2}\cdots(n_1+\cdots+n_r)^{-s_r}$$

This can be regarded as a zeta-function of the root system of type C_r, which is NOT simply-laced.

$\Delta_+ = \Delta_{l+} \cup \Delta_{s+}$ (l: long roots, s: short roots)

$\mathbf{s}_l = (s_\alpha)_{\alpha\in\Delta_+}$, with $s_\alpha = 0$ for any $\alpha \in \Delta_{s+}$

Then we find $\zeta_r(\mathbf{s}_l, C_r) = \zeta_{EHZ,r}(\mathbf{s})$

(It is also possible to understand $\zeta_{EHZ,r}(\mathbf{s})$ as a zeta-function of the root system of type A_r; cf. Komori-M-Tsumura, Math. Z. **268** (2011))

Special values (at positive integer points)

The study of special values is important for both Witten zeta-functions and Euler-Hoffman-Zagier sums

$\Rightarrow$ How are the special values of zeta-functions of root systems?

Recall : Witten's volume formula (coonected with the volumes of certain moduli spaces appearing in quantum gauge theory)

$$\zeta_W(2k, \mathfrak{g}) = C_W(2k, \mathfrak{g})\pi^{2kn}$$

for $k \in \mathbb{N}$, where $n = |\Delta_+|$ and $C_W(2k, \mathfrak{g}) \in \mathbb{Q}$

What is the explicit value of $C_W(2k, \mathfrak{g})$?

1. A_1 case (the Riemann zeta)

$\zeta(2) = \pi^2/6,\ \zeta(4) = \pi^4/90, \ldots$

That is, $C_W(2, A_1) = 1/6,\ C_W(4, A_1) = 1/90, \ldots$

2. A_2 case (the Tornheim sum)

$\zeta_W(s, A_2) = 2^s \zeta_2((s, s, s), \Delta(A_2))$

and Mordell proved $\zeta_2((2, 2, 2), A_2) = \pi^6/2835$

$\Rightarrow C_W(2, A_2) = 4/2835$

Algorithm of computing $C_W(2k, \mathfrak{g})$ for general case:

Szenes (1998), Gunnells-Sczech (2003),

Our approach (Around $2005 -$)

$W = W(\Delta)$: The Weyl group of Δ

Define

$$S(\mathbf{s},\Delta)=\sum_{w\in W}\left(\prod_{\alpha\in\Delta_+\cap w\Delta_-}(-1)^{-s_\alpha}\right)\zeta_r(w^{-1}\mathbf{s},\Delta),$$

where $w^{-1}\mathbf{s}$ is defined by $(\sigma_\beta\mathbf{s})_\alpha = s_{\sigma_\beta\alpha}$ for the reflection σ_β with respect to (the hyperplane orthogonal to) β

(If $\sigma_\beta\alpha\in\Delta_-$, then we understand that $s_{\sigma_\beta\alpha}=s_{-\sigma_\beta\alpha}$)

$S(\mathbf{s},\Delta)$ is the "Weyl group symmetric linear combination" of $\zeta_r(\mathbf{s},\Delta)$

$\Rightarrow$ can be expressed as a certain multiple integral involving a product of Lerch-type zeta-functions

$\Rightarrow$ When $\mathbf{s} = \mathbf{k} = (k_\alpha)_{\alpha\in\Delta_+}$ $(k_\alpha \in \mathbb{N})$, we have

$$S(\mathbf{k}, \Delta) = (-1)^n \left(\prod_{\alpha\in\Delta_+} \frac{(2\pi\sqrt{-1})^{k_\alpha}}{k_\alpha!} \right) B_{\mathbf{k}}(\Delta)$$

($B_{\mathbf{k}}(\Delta)$: a "root-theoretic" generalization of Bernoulli numbers)

In particular, if $k_\alpha = k_\beta$ whenever $||\alpha|| = ||\beta||$, then $w^{-1}\mathbf{k} = \mathbf{k}$, so

$$\begin{aligned} S(2\mathbf{k}, \Delta) &= \sum_{w\in W} \left(\prod_{\alpha\in\Delta_+\cap w\Delta_-} (-1)^{-2k_\alpha} \right) \zeta_r(w^{-1}2\mathbf{k}, \Delta) \\ &= \sum_{w\in W} \zeta_r(2\mathbf{k}, \Delta) = |W|\zeta_r(2\mathbf{k}, \Delta), \quad \text{and hence} \end{aligned}$$

$$\zeta_r(2\mathbf{k}, \Delta) = \frac{(-1)^n}{|W|} \left(\prod_{\alpha\in\Delta_+} \frac{(2\pi\sqrt{-1})^{2k_\alpha}}{(2k_\alpha)!} \right) B_{2\mathbf{k}}(\Delta)$$

$B_{\mathbf{k}}(\Delta)$ can be calculated via its generating function

$$F(\mathbf{t},\Delta)=\sum_{\mathbf{k}} B_{\mathbf{k}}(\Delta)\prod_{\alpha\in\Delta_+}\frac{t_\alpha^{k_\alpha}}{k_\alpha!}\qquad (\mathbf{t}=(t_\alpha)_{\alpha\in\Delta_+})$$

Example:

$$F(t,A_1)=\frac{t}{e^t-1}\qquad \text{(classical case)}$$

$$F((t_1,t_2,t_3),A_2) = \frac{t_1t_2t_3e^{t_1+t_2}(e^{t_3-t_1-t_2}-1)}{(e^{t_1}-1)(e^{t_2}-1)(e^{t_3}-1)(t_3-t_1-t_2)}$$

Now we consider our main target : type G_2

Type G_2 :

$\Psi = \{\alpha_1, \alpha_2\}$

$\Delta_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$

$P_{++} = \{m\lambda_1 + n\lambda_2 \mid m, n \in \mathbb{N}\}$

Therefore

$$\begin{aligned}\zeta_2(\mathbf{s}, G_2) &= \zeta_2((s_1, s_2, s_3, s_4, s_5, s_6), G_2)\\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-s_1} n^{-s_2} (m+n)^{-s_3} (m+2n)^{-s_4}\\ &\qquad \times (m+3n)^{-s_5} (2m+3n)^{-s_6}\end{aligned}$$

Now compute the generating function $F((t_1, t_2, t_3, t_4, t_5, t_6), G_2)$ of the Bernoulli numbers $B_{\mathbf{k}}(G_2)$:

$$
\begin{aligned}
F((t_1,t_2,t_3,t_4,t_5,t_6),G_2) &= t_1t_2t_3t_4t_5t_6 \\
\times\Bigg(&\frac{1}{(e^{t_1}-1)(e^{t_2}-1)}(t_1+t_2-t_3)^{-1}(t_1+2t_2-t_4)^{-1} \\
&\qquad\times(t_1+3t_2-t_5)^{-1}(2t_1+3t_2-t_6)^{-1} \\
+&\frac{1}{(e^{t_1}-1)(e^{t_3}-1)}(t_1+t_2-t_3)^{-1}(t_1-2t_3+t_4)^{-1} \\
&\qquad\times(2t_1-3t_3+t_5)^{-1}(t_1-3t_3+t_6)^{-1} \\
+&\frac{e^{t_1/2+t_4/2}+1}{2(e^{t_1}-1)(e^{t_4}-1)}(t_1/2+t_2-t_4/2)^{-1}(t_1/2-t_3+t_4/2)^{-1} \\
&\qquad\times(t_1/2-3t_4/2+t_5)^{-1}(t_1/2+3t_4/2-t_6)^{-1} \\
-&\frac{e^{2t_1/3+t_5/3}+e^{t_1/3+2t_5/3}+1}{3(e^{t_1}-1)(e^{t_5}-1)}(t_1/3-t_4+2t_5/3)^{-1} \\
&\times(t_1/3+t_2-t_5/3)^{-1}(2t_1/3-t_3+t_5/3)^{-1}(t_1+t_5-t_6)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{e^{t_1/3+t_6/3}+e^{2t_1/3+2t_6/3}+1}{3(e^{t_1}-1)(e^{t_6}-1)}(t_1+t_5-t_6)^{-1}\\
&\qquad\times(t_1/3+t_4-2t_6/3)^{-1}(2t_1/3+t_2-t_6/3)^{-1}(t_1/3-t_3+t_6/3)^{-1}\\
&-\frac{e^{t_2}}{(e^{t_2}-1)(e^{t_3}-1)}(t_1+t_2-t_3)^{-1}(t_2+t_3-t_4)^{-1}\\
&\qquad\times(2t_2+t_3-t_5)^{-1}(t_2+2t_3-t_6)^{-1}\\
&-\frac{e^{t_2}}{(e^{t_2}-1)(e^{t_4}-1)}(t_1+2t_2-t_4)^{-1}(t_2+t_3-t_4)^{-1}\\
&\qquad\times(t_2+t_4-t_5)^{-1}(t_2-2t_4+t_6)^{-1}\\
&+\frac{e^{t_2}}{(e^{t_2}-1)(e^{t_5}-1)}(t_1+3t_2-t_5)^{-1}(2t_2+t_3-t_5)^{-1}\\
&\qquad\times(t_2+t_4-t_5)^{-1}(3t_2-2t_5+t_6)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{e^{t_2/2+t_6/2}+e^{t_2}}{2(e^{t_2}-1)(e^{t_6}-1)}(t_1+3t_2/2-t_6/2)^{-1} \\
&\qquad\times(t_2/2+t_3-t_6/2)^{-1}(t_2/2-t_4+t_6/2)^{-1}(3t_2/2-t_5+t_6/2)^{-1} \\
&-\frac{e^{t_4}}{(e^{t_3}-1)(e^{t_4}-1)}(t_2+t_3-t_4)^{-1}(t_1-2t_3+t_4)^{-1} \\
&\qquad\qquad\times(t_3-2t_4+t_5)^{-1}(t_3+t_4-t_6)^{-1} \\
&+\frac{e^{t_5}}{2(e^{t_3}-1)(e^{t_5}-1)}(t_2+t_3/2-t_5/2)^{-1}(t_3/2-t_4+t_5/2)^{-1} \\
&\qquad\qquad\times(t_1-3t_3/2+t_5/2)^{-1}(3t_3/2+t_5/2-t_6)^{-1} \\
&+\frac{e^{t_6}}{(e^{t_3}-1)(e^{t_6}-1)}(3t_3+t_5-2t_6)^{-1}(t_2+2t_3-t_6)^{-1} \\
&\qquad\qquad\times(t_3+t_4-t_6)^{-1}(t_1-3t_3+t_6)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{e^{t_5}}{(e^{t_4}-1)(e^{t_5}-1)}(t_2+t_4-t_5)^{-1}(t_3-2t_4+t_5)^{-1}\\
&\qquad\qquad\qquad\times(t_1-3t_4+2t_5)^{-1}(3t_4-t_5-t_6)^{-1}\\
&-\frac{e^{t_4}}{(e^{t_4}-1)(e^{t_6}-1)}(t_1+3t_4-2t_6)^{-1}(t_3+t_4-t_6)^{-1}\\
&\qquad\qquad\qquad\times(t_2-2t_4+t_6)^{-1}(3t_4-t_5-t_6)^{-1}\\
&+\frac{e^{t_5}+e^{2t_5/3+2t_6/3}+e^{t_5/3+t_6/3}}{3(e^{t_5}-1)(e^{t_6}-1)}(t_1+t_5-t_6)^{-1}\\
&\quad\times(t_3+t_5/3-2t_6/3)^{-1}(t_4-t_5/3-t_6/3)^{-1}(t_2-2t_5/3+t_6/3)^{-1}\Bigg)
\end{aligned}
$$

From this explicit form of the generating function, we can calculate the special values of $\zeta_2(\mathbf{s}, G_2)$

$$\zeta_2((2,2,2,2,2,2),G_2) = \frac{23}{297904566960}\pi^{12}$$

$$\zeta_2((4,4,4,4,4,4),G_2) = \frac{8165653}{1445838676129559305994400000}\pi^{24}$$

$$\zeta_2((6,6,6,6,6,6),G_2)$$
$$= \frac{55940539974690617}{131888156302530666544150214880458495963616000000}\pi^{36}$$

Also (since our condition is $k_\alpha = k_\beta$ for $||\alpha|| = ||\beta||$) we have

$$\zeta_2((2,4,4,4,2,2),G_2) = \frac{467}{213955059990672000}\pi^{18}$$

How to evaluate the values at “odd” integer points?

Recall :

$$S(\mathbf{k}, G_2) = \sum_{w \in W} \left(\prod_{\alpha \in \Delta_+ \cap w\Delta_-} (-1)^{-k_\alpha} \right) \zeta_r(w^{-1}\mathbf{k}, G_2)$$

When k_α is odd, the signature part appears.

For G_2, it is well-known that $|W| = 12$, and it is easy to see that $|\Delta_+ \cap w\Delta_-|$ is odd for 6 elements of W and is even for the other 6 elements of W.

Therefore, when $\mathbf{k} = (k, k, k, k, k, k)$ for some odd positive integer k, all terms on the right-hand side of the above equation are cancelled, and we obtain NO information.

However, we can show, for example, the following "functional relation" :

$$\begin{aligned} &\zeta_2((2,s,1,1,1,1),G_2)+\zeta_2((2,1,s,1,1,1),G_2) \\ &-\zeta_2((1,1,1,s,2,1),G_2)+\zeta_2((1,1,1,s,1,2),G_2) \\ &-\zeta_2((1,1,s,1,1,2),G_2)+\zeta_2((1,s,1,1,2,1),G_2) \\ &=\frac{1}{9}\zeta(2)\zeta(s+4)-\frac{109}{648}\zeta(s+6) \end{aligned}$$

Putting $s=1$, we obtain

$$\zeta_2((2,1,1,1,1,1),G_2)=\frac{1}{18}\zeta(2)\zeta(5)-\frac{109}{1296}\zeta(7)$$

Note : $2+1+1+1+1+1$ is odd

Similarly,

$$\zeta_2((2,1,1,1,2,2),G_2) = -\frac{187}{324}\zeta(2)\zeta(7) + \frac{11149}{11664}\zeta(9)$$

$$\zeta_2((4,2,2,1,1,1),G_2) = \frac{1}{18}\zeta(4)\zeta(7) + \frac{595}{648}\zeta(2)\zeta(9) - \frac{73201}{46656}\zeta(11)$$

$$\zeta_2((2,4,4,3,3,3),G_2) = \frac{1}{8}\zeta(4)\zeta(15) + \frac{281221}{23328}\zeta(2)\zeta(17) - \frac{11177971}{559872}\zeta(19)$$

Note : $2+1+1+1+2+2$, $4+2+2+1+1+1$, $2+4+4+3+3+3$ are all odd

Underlying reason :

$I \subset \{1, 2, \dots, r\}, \quad I \neq \emptyset$

$\Psi_I = \{\alpha_i \mid i \in I\} \subset \Psi$

V_I : the linear space spanned by Ψ_I

$\Delta_I = \Delta \cap V_I$

$W^I = \{w \in W \mid \Delta_{I+}^{\vee} \subset \Delta_+^{\vee}\}$

and define

$$S(\mathbf{s}, I, \Delta) = \sum_{w \in W^I} \left(\prod_{\alpha \in \Delta_+ \cap w\Delta_-} (-1)^{-s_\alpha} \right) \zeta_r(w^{-1}\mathbf{s}, \Delta)$$

Then we can show the following theorem:

Theorem : If there exist $w_1 \in W$ and $\mathbf{s}$ satisfying

- $s_\alpha \in \mathbb{Z}$ for $\alpha \in \Delta_+ \cap w_1\Delta_-$,
- $w_1^{-1}\mathbf{s} = \mathbf{s}$,
- $w_1^{-1}W^I = w_1 W^I$,

and moreover

$$\left(\prod_{\alpha \in \Delta_+ \cap w_1 \Delta_-} (-1)^{s_\alpha} \right) = -1,$$

then we have

$$S(\mathbf{s}, I, \Delta) = \sum_{w \in W^I \setminus w_1 W^I} \left(\prod_{\alpha \in \Delta_+ \cap w \Delta_-} (-1)^{-s_\alpha} \right) \zeta_r(w^{-1}\mathbf{s}, \Delta)$$

In the case of G_2, we can apply this theorem with

$I = \{2\}$, $w_1 = -\sigma_{\alpha_1}$, and

$\mathbf{s} = (s_1, s_2, s_2, s_4, s_5, s_5)$ with odd s_4 (necessary for $w_1\mathbf{s} = \mathbf{s}$ and the signature condition)

$$\Rightarrow \quad S(\mathbf{s}, \{2\}, G_2) = \zeta_2(\mathbf{s}, G_2) + (-1)^{s_1}\zeta_2(\mathbf{s}, G_2)$$

$$\Rightarrow \quad \zeta_2(\mathbf{s}, G_2) = (1/2)S(\mathbf{s}, \{2\}, G_2)$$

for $\mathbf{s} = (s_1, s_2, s_2, s_4, s_5, s_5)$ with even s_1 and odd s_4

(therefore $s_1 + s_2 + s_2 + s_4 + s_5 + s_5$ is odd)

Parity result : Some multiple zeta value whose weight and depth are of different parity can be written in terms of multiple zeta values of lower depth

• Euler proved that $\sum_{m\geq 1}\sum_{n\geq 1} m^{-p}(m+n)^{-q}$ can be written as a linear combination of $\zeta(j)$ $(j \geq 2)$ with rational coefficients, when $p+q$ is odd

• Similar result for zeta values of type A_2 (Tornheim)

• Similar result for zeta values of type B_2 (Tsumura)

How is the case of G_2?

$\Rightarrow$ It seems that we need some modification, because

$$\begin{aligned}\zeta_2((1,2,1,1,1,1),G_2) &= \frac{1}{2}\zeta(2)\zeta(5) - \frac{109}{16}\zeta(7) \\ &\quad + \frac{81}{8}L(1,\chi_3)L(6,\chi_3)\end{aligned}$$

(χ_3 : the primitive Dirichlet character mod 3)

Problem 1 : Can the values $\zeta_2((k_1, k_2, k_3, k_4, k_5, k_6), G_2)$ be expressed in terms of $\zeta(j)$ and $L(j, \chi_3)$ when $k_1 + k_2 + k_3 + k_4 + k_5 + k_6$ is odd?

cf. It is known that they are expressed in terms of double polylogarithms (Zhao, 2010). Okamoto (2012) proved that they can be expressed by $\zeta(j)$, $L(j, \chi_3)$, $S_r(j/l)$ and $C_r(j/l)$ with $l = 4, 12$ and $0 < j < l$, $(j, l) = 1$, where $S_r(x) = \sum_{m \geq 1} \sin(2\pi m x) m^{-r}$ and $C_r(x) = \sum_{m \geq 1} \cos(2\pi m x) m^{-r}$ (Clausen functions)

Problem 2 : When the values $\zeta_2((k_1, k_2, k_3, k_4, k_5, k_6), G_2)$ be expressed only in terms of $\zeta(j)$?