指数型不定方程式 $a^x + lb^y = c^z$ について

寺井 伸浩 (足利工業大学)

Conjecture 1. (**Generalized Jeśmanowicz' conjecture.**) (cf. Terai [T1], [T2])
Let a, b, c, p, q, r be fixed positive integers satisfying $a^p + b^q = c^r$ with $p,\ q\ ,r \geq 2$ and $\gcd(a, b, c) = 1$. *Then the Diophantine equation*

$$a^x + b^y = c^z \tag{1}$$

has only the positive integer solution $(x, y, z) = (p, q, r)$ except for three cases (taking $a < b$), where (1) has only the following solutions, respectively:

$$\begin{aligned} (a,\ b,\ c) &= (2,\ 2^k - 1,\ 2^k + 1), & (x,\ y,\ z) &= (1,\ 1,\ 1),\ (k+2,\ 2,\ 2), \\ (a,\ b,\ c) &= (2,\ 7,\ 3), & (x,\ y,\ z) &= (1,\ 1,\ 2),\ (5,\ 2,\ 4); \\ (a,\ b,\ c) &= (1,\ 2,\ 3), & (x,\ y,\ z) &= (m,\ 1,\ 1),\ (n,\ 3,\ 2); \end{aligned}$$

where $m,\ n$ are arbitrary and k is a positive integer with $k \geq 2$.

Conjecture 2. (**Le's conjecture.**) (cf. Le [Le])
Equation (1) has at most one solution $(x,\ y,\ z)$ with $\gcd(a, b, c) = 1$ *and* $\min(x,\ y,\ z) \geq 2$.

Conjecture 3. (**Generalized Fermat's conjecture.**)
Equation (1) has no positive integer solutions (x, y, z, a, b, c) with $\gcd(a, b, c) = 1$ *and* $\min(x,\ y,\ z) \geq 3$.

Lemma 1. *Let l be an odd prime, a odd and b even. The positive integer solutions of the equation $a^2 + lb^2 = c^2$ with* $\gcd(a, b) = 1$ *are given by*

$$a = \pm(u^2 - lv^2), \quad b = 2uv, \quad c = u^2 + lv^2,$$

where u, v are positive integers such that $\gcd(u, v) = 1$ *and* $u \not\equiv v \pmod 2$.

Lemma 2. (Adachi[A]). *Let l be an odd prime with* $l \equiv 3 \pmod 4$, *and a, b relatively prime integers of opposite parity. Then the implication*

$$a^2 + lb^2 = c^l \Rightarrow a + b\sqrt{-l} = (u + v\sqrt{-l})^l$$

holds for some integers u and v. In particular, when $u = 1$ and $v = m$, we obtain

$$a = -\sum_{j=0}^{(l-1)/2} \binom{l}{2j} m^{2j}(-l)^j > 0, \quad b = -m \sum_{j=0}^{(l-1)/2} \binom{l}{2j+1} m^{2j}(-l)^j > 0, \quad c = lm^2 + 1 \tag{2}$$

with m even.

Lemma 3. (Nagell[N], Theorem 117). *Let l be a prime with $l \equiv 3 \pmod 8$. The Diophantine equation*

$$x^4 - y^4 = lz^2$$

has no solutions in positive integers x, y and z.

Let α_1 and α_2 be real algebraic numbers with α_1 and α_2. We consider the linear form

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1,$$

where b_1 and b_2 are positive integers. Define $b' = b_1/\log \alpha_2 + b_2/\log \alpha_1$.

Lemma 4. (Laurent [La]). *Let Λ be given as above, with $\alpha_1 > 1$ and $\alpha_2 > 1$. Suppose that α_1 and α_2 are multiplicatively independent. Then*

$$\log|\Lambda| \geq -25.2\ (\max\{\log b' + 0.38,\ 10\})^2 \log \alpha_1 \log \alpha_2.$$

Theorem. *Let l be a prime with $l \equiv 3 \pmod 8$. Let a, b, c be as in (2) satisfying $2||m$ and $a^2 > \sqrt{l}\,b$. Suppse that*

$$\frac{2}{l+1}(m^2 + l) \geq 2521 \log(lm^2 + 1).$$

Then the Diophantine equation

$$a^x + lb^y = c^z \tag{3}$$

has only the positive integer solution $(x, y, z) = (2, 2, l)$

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