LIFTS TO SIEGEL MODULAR FORMS OF HALF-INTEGRAL WEIGHT AND THE GENERALIZED MAASS RELATIONS (RESUME).

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The purpose of this talk is to explain the lift from two elliptic modular forms to Siegel modular forms of half-integral weight of degree $2n-2$:

$$S^{+}_{k-\frac{1}{2}} \times S^{+}_{k-n+\frac{1}{2}} \to S^{+(2n-2)}_{k-\frac{1}{2}} \qquad (k: \text{ even}).$$

0. Notation

$$\begin{aligned}
\mathfrak{H}_n &:= \{\tau \in Sym_n(\mathbb{C}) \,|\, Im(\tau) > 0\} \qquad \text{(Siegel upper half space)},\\
\mathrm{Sp}_n(K) &:= \{M \in M_{2n}(K) \,|\, M J_n {}^t M = J_n\} \qquad \text{(symplectic group)},\\
& \quad \text{where } J_n := \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \text{ and where } K \text{ is a commutative ring},\\
\Gamma_n &:= \mathrm{Sp}_n(\mathbb{Z}),\\
\mathrm{Sym}^*_n &:= \text{the set of all half-integral symmetric matrices of size } n,\\
e(x) &:= e^{2\pi\sqrt{-1}\, tr(x)} \text{ for symmetric matrix } x,\\
M \cdot \tau &:= (A\tau + B)(C\tau + D)^{-1} \qquad \left(M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},\ \tau \in \mathfrak{H}_n\right).
\end{aligned}$$

We remark $\mathrm{Sp}_1(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$ and $\mathfrak{H}_1$ is the Poincare upper half plane.

$$\begin{aligned}
S^n_k &:= \text{Siegel cusp forms of weight } k \text{ of degree } n,\\
S^+_{k-\frac{1}{2}} &:= \text{cusp forms in the Kohnen plus space of weight } k-\tfrac{1}{2},\\
S^{+(n)}_{k-\frac{1}{2}} &:= \text{cusp forms in the generalized plus space of weight } k-\tfrac{1}{2} \text{ of degree } n,\\
J^{(n)}_{k,m} &:= \text{Jacobi forms of weight } k \text{ of index } m \text{ of degree } n.
\end{aligned}$$

1. The Saito-Kurokawa lift

1.1. Definition of Siegel modular forms.

Definition 1. *Let F be a holomorphic function on $\mathfrak{H}_n$.*
Such F is called a Siegel modular form of weight k of degree n, if F satisfies

$$F|_k M \;=\; F$$

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for any $M \in \Gamma_n$. Here

$$(F|_k M)(\tau) \quad := \quad \det(C\tau + D)^{-k} F(M \cdot \tau)$$

for $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and $\tau \in \mathfrak{H}_n$.

If $n = 1$, then F is required to satisfy the so-called cusp condtion.

We put

$$\begin{aligned} M_k^n &:= \{\text{Siegel modular form of weight } k \text{ of degree } n\} \\ S_k^n &:= \{\text{Siegel cusp form in } M_k^n\} \end{aligned}$$

1.2. **Fourier expansion.** Let $F \in M_k^n$, then F has the Fourier expansion:

$$F(\tau) \quad = \sum_{\substack{N \in Sym_n^* \\ N \geq 0}} A(N)\, e(N\tau) \qquad (\tau \in \mathfrak{H}_n).$$

We remark $e(N\tau) = e(tr(N\tau)) = e\left(\sum_{1 \leq i \leq j \leq n} n_{i,j}\tau_{i,j}\right)$ for $N = (\frac{1}{1+\delta_{i,j}} n_{i,j})$ and $\tau = (\tau_{i,j})$.

A $F \in M_k^n$ is called a Siegel cusp form if F satisfies the condition:

$$A(N) = 0 \text{ unless } N > 0.$$

1.3. **Maass relation of degree** 2**.** Let $F(\tau) = \sum_N A(N)\, e(N\tau) \in M_k^2$.

The Maass relation is the following relation among Fourier coefficients:

$$\boxed{A\left(\begin{pmatrix} n & \frac{1}{2}r \\ \frac{1}{2}r & m \end{pmatrix}\right) \quad = \sum_{d|(n,m,r)} d^{k-1} A\left(\begin{pmatrix} \frac{nm}{d^2} & \frac{r}{2d} \\ \frac{r}{2d} & 1 \end{pmatrix}\right)}\,.$$

The so-called Maass space (or Maass Spezialscher) is denoted by

$$M_k^{2\,Maass} \quad := \quad \left\{F \in M_k^2 \,|\, F \text{ satisfies the Maass relation}\right\}.$$

1.4. **The Fourier-Jacobi expansion.** Let $F \in M_k^{n+r}$. The expansion

$$F\left(\begin{pmatrix} \tau & z \\ {}^t z & \omega \end{pmatrix}\right) \quad = \sum_{\mathcal{M} \in Sym_r^*} \phi_{\mathcal{M}}(\tau, z)\, e(\mathcal{M}\omega).$$

is called the Fourier-Jacobi expansion of F. Here $\phi_{\mathcal{M}}$ is a Jacobi form of weight k of index $\mathcal{M}$ of degree n (i.e. $\phi_{\mathcal{M}}$ satisfies a certain transformation formula, see below.)

Definition 2 (Jacobi forms of matrix index)**.** *Let ϕ be a holomorphic function on $\mathfrak{H}_n \times M_{n,r}(\mathbb{C})$.*

Such ϕ is called a Jacobi form of weight k of index $\mathcal{M}$ of degree n, if ϕ satisfies the transformation formula:

$$(\phi(\tau, z)\, e(\mathcal{M}\omega))\,|_k\gamma \quad = \quad \phi(\tau, z)\, e(\mathcal{M}\omega)$$

for any $\gamma \in \Gamma^J_{n,r}$. *Here we put*

$$\Gamma^J_{n,r} \quad := \quad \left\{ \begin{pmatrix} * & 0 & * & * \\ * & 1_r & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & 1_r \end{pmatrix} \in \Gamma_{n+r} \right\}.$$

If $n = 1$, *then* ϕ *is required to satisfy the cusp condtion.*

We put

$$J^{(n)}_{k,\mathcal{M}} \quad := \quad \{\text{Jacobi forms of weight } k \text{ of index } \mathcal{M} \text{ of degree } n\}.$$

1.5. **Maass relation of degree 2 (Fourier-Jacobi coefficients version).** Let

$$F(\left(\begin{smallmatrix} \tau & z \\ {}^t z & \omega \end{smallmatrix}\right)) = \sum_{m=0}^{\infty} \phi_m(\tau, z)\, e(m\omega) \in M^2_k$$

be the Fourier-Jacobi expansion of F. (Note $\phi_m \in J^{(1)}_{k,m}$.)

A F satisfies the Maass relation, iff

$$\boxed{\phi_m = \phi_1 | V_m \qquad \text{for any } m \geq 1}.$$

Here $V_m : J^{(1)}_{k,l} \to J^{(1)}_{k,ml}$ is <u>the index-shift operator</u> of Jacobi forms which is defined by

$$(\phi|V_m)(\tau, z) \quad := \quad m^{k-1} \sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1 \backslash M_2(\mathbb{Z}) \\ ad-bc=m}} (c\tau + d)^{-k} e\left(ml \frac{cz^2}{c\tau + d}\right) \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{mz}{c\tau + d}\right)$$

for $\phi \in J^{(1)}_{k,l}$, $(\tau, z) \in \mathfrak{H}_1 \times \mathbb{C}$. We note that $V_1 = id$ and $(\phi|V_m)(\tau, 0) = \phi(\tau, 0)|T(m)$ with the usual Hecke operator $T(m)$ which acts on the space of elliptic modular forms.

1.6. **Saito-Kurokawa lift.** Let k be an even integer. Then we obtain the composition of the isomorphisms:

$$M^1_{2k-2} \cong M^+_{k-\frac{1}{2}} \cong J^{(1)}_{k,1} \cong M^{2\,Maass}_k.$$

Here $M^+_{k-\frac{1}{2}}$ is the Kohnen plus-space (see below for this definiton). The first isomorphism is the Shimura correspondence (which is also true for odd integer k).

If $\phi \in J^{(1)\,cusp}_{k,1}$, then

$$F\left(\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right)\right) \quad = \quad \sum_{m=1}^{\infty} (\phi|V_m)(\tau, z)\, e(m\omega) \in S^{2\,Maass}_k.$$

Theorem 1.1 (Saito-Kurokawa lift)**.** *Let* $f \in S^1_{2k-2}$ *be a normalized Hecke eigenform. Then there exists* $F \in S_k^{2\,Maass}$*, such that* F *is a Hecke eigenform with*

$$L(s,F,sp) \;=\; \zeta(s-k+1)\zeta(s-k+2)L(s,f),$$

where $L(s,F,sp)$ *is the spinor L-function of* F*.*

This lift was conjectured by H.Saito and Kurokawa, independently, and proved by Maass, Andrianov, Zagier.

2. The Ikeda lift

For the detail of the Ikeda lift the reader is referred to the original paper by T.Ikeda (2001). These lifts had been conjectured by Duke-Imamoglu and Ibukiyama, independently.

2.1. **The Siegel series.** For $B \in Sym^*_{2n}$ we put

$$b_p(B,s) \;:=\; \sum_{R \in Sym_{2n}(\mathbb{Q}_p)/Sym_{2n}(\mathbb{Z}_p)} \psi_p(tr(BR))\, p^{-ord_p(\det(D))s}, \qquad (Re(s)>0),$$

where $\psi_p(x) := e((-1)^{n+1}x)$ for $x \in \mathbb{Z}[p^{-1}]$ and D is determined by the identity $C^{-1}D = R$ with a symmetric coprime pair $\{C, D\}$.

It is known that there exists a polynomial $F_p(B,X)$ which satisfies

$$F_p(B,p^{-s}) \;=\; b_p(B,s)\gamma(B,p^{-s})^{-1}$$

with a certain elementary rational polynomial $\gamma(B,X)$. We put

$$\tilde{F}_p(B,X) \;:=\; p^{l(B)}F_p(B,p^{-n-\frac{1}{2}}X)$$

with a certain number $l(B)$. Then it is known the fact $\tilde{F}_p(B,X) \in \mathbb{C}[X+X^{-1}]$.

We write

$$(-1)^n \det(2B) \;=\; D_B f_B^2,$$

where D_B is the discriminant of $\mathbb{Q}(\sqrt{(-1)^n\det(2B)}))/\mathbb{Q}$, and f_B is a positive-integer. If $f_B = 1$, then $\tilde{F}_p(B,X) = 1$.

2.2. **The Ikeda lift.** We assume k is an even integer. Let $g(z) = \sum_m c(m)e(mz) \in S^+_{k-n+\frac{1}{2}}$ be a Hecke eigenform in the Kohnen plus-space. Let $\lambda(p)$ be the eigenvalue of g for the Hecke operator $T^+_1(p^2)$. The parameters $\{\beta^{\pm}_p\}$ are determined through the identity

$$1-\lambda(p)T+p^{2k-2n-1}T^2 \;=\; (1-\beta_p p^{k-n-\frac{1}{2}}T)(1-\beta_p^{-1}p^{k-n-\frac{1}{2}}T).$$

We put

$$A(B) \;=\; c(|D_B|)f_B^{k-n-\frac{1}{2}}\prod_{p|f_B}\tilde{F}_p(B,\beta_p)$$

and

$$I(g)(\tau) \ := \sum_{\substack{B\in Sym^*_{2n}\\ B>0}} A(B)\, e(B\tau) \qquad (\tau \in \mathfrak{H}_{2n}).$$

Theorem 2.1 (Ikeda)**.** *The above $I(g)$ is a Siegel cusp form of weight k of degree $2n$. Moreover, the form $I(g)$ is an eigenform for any Hecke operator and the standard L-function $L(s, I(g), st)$ of F satisfies*

$$L(s, G, st) \ = \ \zeta(s) \prod_{i=1}^{2n} L(s+k-i, g).$$

3. Siegel modular forms of half-integral weight

We put

$$\theta(\tau) \ := \sum_{p\in\mathbb{Z}^n} e(p\tau\, {}^t p) \qquad (\tau \in \mathfrak{H}_n).$$

A holomorphic function F on $\mathfrak{H}_n$ is said to be a Siegel modular form of weight $k - \frac{1}{2}$, if F satisfies the transformation formula

$$F(M\cdot\tau) \ = \ \left(\frac{\theta(M\cdot\tau)}{\theta(\tau)}\right)^{2k-1} F(\tau) \qquad \text{for any } M \in \Gamma_0^{(n)}(4).$$

The generalized plus space $S^{+(n)}_{k-\frac{1}{2}}$ consists of all Siegel cusp forms F of weight $k - \frac{1}{2}$ which satisfy

$$A_F(N) = 0 \qquad \text{unless} \qquad N \equiv (-1)^{k+1}\lambda\, {}^t\lambda \mod 4 \text{ with some } \lambda \in \mathbb{Z}^n,$$

where $A_F(N)$ is the N-th Fourier coefficient of F.

4. Main theorem

Conjecture 1 (Ibukiyama-H. 2005)**.** *Let k be an integer. Let $f \in S^+_{k-\frac{1}{2}}$ and $g \in S^+_{k-\frac{3}{2}}$ be Hecke eigenforms in the Kohnen plus-spaces. Then there exists $\mathcal{F}_{f,g} \in S^{+(2)}_{k-\frac{1}{2}}$, such that the form $\mathcal{F}_{f,g}$ is a Hecke eigenform with the (modified) Zhuravlev L-function*

$$L(s, \mathcal{F}_{f,g}) \ = \ L(s, f) L(s-1, g).$$

Theorem 4.1 (H.)**.** *Let k be an <u>even</u> integer. Let $f \in S^+_{k-\frac{1}{2}}$ and $g \in S^+_{k-n+\frac{1}{2}}$ be Hecke eigenforms in the Kohnen plus-spaces. Then there exists $\mathcal{F}_{f,g} \in S^{+(2n-2)}_{k-\frac{1}{2}}$. Under the assumption $\mathcal{F}_{f,g} \not\equiv 0$, the form $\mathcal{F}_{f,g}$ is a Hecke eigenform with the (modified) Zhuravlev L-function*

$$L(s, \mathcal{F}_{f,g}) \ = \ L(s, f) \prod_{i=1}^{2n-3} L(s-i, g).$$

The construction of the lift $\mathcal{F}_{f,g}$ (which is suggested by T.Ikeda):

$$\begin{array}{ccccc}
 & I(g) \in S_k^{2n} & & & \\
 & \Big\downarrow \text{1st F-J} & & & \\
 & \psi_1 \in J_{k,1}^{(2n-1)} & \xrightarrow[\text{E-Z-I}]{} & G \in S_{k-\frac{1}{2}}^{+(2n-1)} & \\
\nearrow \text{Ikeda lift} & & & \Big\downarrow \text{F-J} & \\
 & & & \bigoplus_{m\equiv 0,3 \bmod 4} \phi_m^{(2n-2)} \in \bigoplus_{m\equiv 0,3 \bmod 4} J_{k-\frac{1}{2},m}^{(2n-2)}. & \\
g \in S_{k-n+\frac{1}{2}}^{+} & & & &
\end{array}$$

where F-J = Fourier-Jacobi expansion, E-Z-I = Eichler-Zagier-Ibukiyama correspondence, and we put

$$\mathcal{F}_{f,g}(\tau) \ := \ \frac{1}{6}\int_{\Gamma_0(4)\backslash\mathfrak{H}_1} G\left(\begin{pmatrix}\tau & 0\\ 0 & \omega\end{pmatrix}\right)\overline{f(\omega)}\, Im(\omega)^{k-\frac{5}{2}}\, d\omega.$$

The key of the proof of Theorem 4.1 is to show the following generalized Maass relation.

Theorem 4.2 (A generalized Maass relation)**.** *Let $\phi_m^{(2n-2)}$ be the m-th Fourier-Jacobi coefficient of G in the above diagram. Then we obtain*

$$\begin{aligned}
&\phi_m^{(2n-2)}|(\tilde{V}_{0,2n-2}(p^2),\tilde{V}_{1,2n-3}(p^2),...,\tilde{V}_{2n-2,0}(p^2))\\
= \ & p^{k(2n-3)-2n^2-n+\frac{11}{2}}\left(\phi_{\frac{m}{p^2}}^{(2n-2)}|U_{p^2},\, \phi_m^{(2n-2)}|U_p,\, \phi_{mp^2}^{(2n-2)}\right)\begin{pmatrix}0 & p^{2k-3}\\ p^{k-2} & p^{k-2}\left(\frac{-m}{p}\right)\\ 0 & 1\end{pmatrix}\\
&\times A_{2,2n-1}^{p}(\beta_p)\ diag(1,p^{\frac{1}{2}},p,p^{\frac{3}{2}}...,p^{n-1})
\end{aligned}$$

for any prime p. (The both sides are vectors of forms.) Here

$$\tilde{V}_{i,2n-2-i}(p^2): J_{k-\frac{1}{2},m}^{+(2n-2)} \to J_{k-\frac{1}{2},mp^2}^{+(2n-2)},$$

$$U_{p^j}: J_{k-\frac{1}{2},m}^{+(2n-2)} \to J_{k-\frac{1}{2},mp^{2j}}^{+(2n-2)}$$

are certain index-shift operators of Jacobi forms of half-integral weight of degree $2n-2$, and $A_{2,2n-1}^{p}(\beta_p)$ is a $2\times(2n-1)$-matrix which depends only on the choice of p and $g \in S_{k-n+\frac{1}{2}}^{+}$.