Pullbacks of Hermitian Maass lifts

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1 Notation

K an imaginary quadratic field,
$-D < 0$ the discriminant of K (so that $D \in \mathbb{Z}_{>0}$),
Q_D the set of primes dividing D,
$\mathfrak{o}$ the ring of integers of K,
$\mathfrak{c} \subset \mathfrak{o}$ an integral ideal which is prime to D,
$C = N(\mathfrak{c}) \in \mathbb{Z}_{>0}$ the ideal norm,
χ the Dirichlet character corresponding $K/\mathbb{Q}$,
$\underline{\chi} = \otimes_v \underline{\chi}_v$ the character of $\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times$ determined by χ.
We define a primitive Dirichlet character χ_p by

$$\chi_p(n) = \begin{cases} \chi(m) & \text{if } (n,p) = 1, \\ 0 & \text{if } p \mid n, \end{cases}$$

with

$$m = \begin{cases} n & \bmod D_p = p^{\mathrm{ord}_p(D)}, \\ 1 & \bmod D_p^{-1}D. \end{cases}$$

For $Q \subset Q_D$, we put

$$\chi_Q = \prod_{p \in Q} \chi_p, \quad \chi'_Q = \prod_{p \in Q_D \setminus Q} \chi_p.$$

$\mathfrak{H}$ the complex upper half plane.

2 Hermitian Maass lifts

Let κ be a positive integer and

$$f = \sum_{n>0} a_f(n) q^n \in S_{2\kappa+1}(\Gamma_0(D), \chi)$$

be a normalized Hecke eigenform. For a subset Q of Q_D, there exists a normalized Hecke eigenform $f_Q \in S_{2\kappa+1}(\Gamma_0(D), \chi)$ such that

$$a_{f_Q}(p) = \begin{cases} \chi_Q(p) a_f(p) & \text{if } p \notin Q, \\ \chi'_Q(p) \overline{a_f(p)} & \text{if } p \in Q \end{cases}$$

for all prime p. We put

$$f^{\mathfrak{c}*} = \sum_{Q \subset Q_D} \chi_Q(-C) f_Q.$$

Lemma 1. *The Fourier coefficients of $f^{\mathfrak{c}*}$ are purely imaginary.*

The Fourier coefficient of $f^{\mathfrak{c}*}$ satisfies $a_{f^{\mathfrak{c}*}}(n) = \mathbf{a}_D^{\mathfrak{c}}(n)\alpha_{F_{\mathfrak{c}}}(n)$ with $\mathbf{a}_D^{\mathfrak{c}}(n) = \prod_{p|D}(1+\chi_p(-Cn))$. Using $\alpha_{F_{\mathfrak{c}}}(n)$, we construct the hermitian Maass lifts.

The hermitian upper half space $\mathcal{H}_2$ of degree 2 is defined by

$$\mathcal{H}_2 = \left\{ Z \in \mathrm{M}_2(\mathbb{C}) \left| \frac{1}{2\sqrt{-1}}(Z - {}^t\bar{Z}) > 0 \right.\right\}.$$

Note that $\mathcal{H}_2$ contains $\mathfrak{H} \times \mathfrak{H}$ as the diagonal matrices. Let

$$\mathrm{U}(2,2)(\mathbb{Q}) = \left\{ g \in \mathrm{GL}_4(K) \left| {}^t\bar{g} \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix} \right.\right\}$$

be the unitary group and

$$\Gamma_K^{(2)}[\mathfrak{c}] = \left\{ g \in \mathrm{U}(2,2)(\mathbb{Q}) \left| g \begin{pmatrix} \mathfrak{o} \\ \mathfrak{c} \\ \mathfrak{o} \\ \bar{\mathfrak{c}}^{-1} \end{pmatrix} = \begin{pmatrix} \mathfrak{o} \\ \mathfrak{c} \\ \mathfrak{o} \\ \bar{\mathfrak{c}}^{-1} \end{pmatrix} \right.\right\}$$

be a discrete subgroup of $\mathrm{U}(2,2)(\mathbb{Q})$. We put

$$\Lambda_2^{\mathfrak{c}}(\mathfrak{o}) = \left\{ \begin{pmatrix} n & \alpha \\ \bar{\alpha} & m/C \end{pmatrix} \in \mathrm{M}_2(K) \left| n, m \in \mathbb{Z}, \alpha \in \sqrt{-D}^{-1}\mathfrak{c}^{-1} \right.\right\}$$

and

$$\varepsilon_{\mathfrak{c}}(H) = \max\{m \in \mathbb{Z}_{>0} | m^{-1}H \in \Lambda_2^{\mathfrak{c}}(\mathfrak{o})\}$$

for $H \in \Lambda_2^{\mathfrak{c}}(\mathfrak{o})$. We denote by $\Lambda_2^{\mathfrak{c}}(\mathfrak{o})^+$ the subset of $\Lambda_2^{\mathfrak{c}}(\mathfrak{o})$ of positive definite elements.

We define the hermitian Maass lift $F_{\mathfrak{c}}$ of f which satisfies the Maass relation for $\mathfrak{c}$ by

$$F_{\mathfrak{c}}(Z) = \sum_{H \in \Lambda_2^{\mathfrak{c}}(\mathfrak{o})^+} \left(\sum_{d|\varepsilon(H)} d^{2\kappa+1} \alpha_{F_{\mathfrak{c}}}\left(\frac{CD\det(H)}{d^2}\right)\right) \exp(2\pi\sqrt{-1}\mathrm{Tr}(HZ)).$$

The lift $F_{\mathfrak{c}}$ is a holomorphic function on $\mathcal{H}_2$ and satisfies

$$F_{\mathfrak{c}}(Z) = \det(\gamma)^{\kappa+1} F_{\mathfrak{c}}((AZ+B)(CZ+D)^{-1})(CZ+D)^{-(2\kappa+2)}$$

for $Z \in \mathcal{H}_2$ and

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_K^{(2)}[\mathfrak{c}].$$

We find that

$$F_{\mathfrak{c}}\left(\mathrm{diag}(z_1, z_2)\right) \in S_{2\kappa+2}(\mathrm{SL}_2(\mathbb{Z})) \otimes S_{2\kappa+2}(d(C)^{-1}\mathrm{SL}_2(\mathbb{Z})d(C))$$

with $d(C) = \mathrm{diag}(1, C) \in \mathrm{GL}_2(\mathbb{Q})$. For $g(z) = \sum_{n=1}^{\infty} a_g(n)q^n \in S_{2\kappa+2}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform, we put $g_C(z) = g(z/C) \in S_{2\kappa+2}(d(C)^{-1}\mathrm{SL}_2(\mathbb{Z})d(C))$. We consider the period integral $\langle F_{\mathfrak{c}}|_{\mathfrak{H}\times\mathfrak{H}}, g \times g_C\rangle$ given by

$$\langle F_{\mathfrak{c}}|_{\mathfrak{H}\times\mathfrak{H}}, g \times g_C\rangle =$$
$$\int_{d(C)^{-1}\mathrm{SL}_2(\mathbb{Z})d(C)\backslash\mathfrak{H}} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}} F_{\mathfrak{c}}\left(\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}\right) \overline{g(z_1)g_C(z_2)} y_1^{2\kappa} y_2^{2\kappa} dz_1 dz_2.$$

Here, dz_1, dz_2 are Lebesgue measures, i.e.,

$$\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}, y^{-2}dz) = \frac{\pi}{3}.$$

Lemma 2. *Let $f \in S_{2\kappa+1}(\Gamma_0(D), \chi)$ and $g, g_1, g_2 \in S_{2\kappa+2}(\mathrm{SL}_2(\mathbb{Z}))$ be normalized Hecke eigenforms and $\mathfrak{c}$ be an integral ideal of K which is prime to D.*

1. *$\langle F_{\mathfrak{c}}|_{\mathfrak{H}\times\mathfrak{H}}, g_1 \times (g_2)_C\rangle = 0$ unless $g_1 = g_2$.*
2. *Fix f and g. Then the map*

$$\mathfrak{c} \mapsto \frac{\langle F_{\mathfrak{c}}|_{\mathfrak{H}\times\mathfrak{H}}, g \times g_C\rangle}{\langle g, g\rangle\langle g_C, g_C\rangle} \in \mathbb{Q}(f)\mathbb{Q}(g) \cap \sqrt{-1}\mathbb{R}$$

depends only on the ideal class of $\mathfrak{c}$.

3 Main Theorem

We define the Satake parameter $\{\alpha_{f,p}, \chi(p)\alpha_{f,p}^{-1}\}$ of f by

$$\begin{cases} 1 - a_f(p)X + \chi(p)p^{2\kappa}X^2 = (1 - p^{\kappa}\alpha_{f,p}X)(1 - p^{\kappa}\chi(p)\alpha_{f,p}^{-1}X) & \text{if } p \notin Q_D, \\ \alpha_{f,p} = p^{-\kappa}a_f(p) & \text{if } p \in Q_D, \end{cases}$$

and the Satake parameter $\{\alpha_{g,p}, \alpha_{g,p}^{-1}\}$ of g by

$$1 - a_g(p)X + p^{2\kappa+1}X^2 = (1 - p^{\kappa+1/2}\alpha_{g,p}X)(1 - p^{\kappa+1/2}\alpha_{g,p}^{-1}X).$$

Then, the Ramanujan conjecture (proved by Deligne) states that

$$|\alpha_{f,p}| = |\alpha_{g,p}| = 1 \quad \text{for all } p.$$

We put

$$A_p = \begin{cases} \begin{pmatrix} \alpha_{f,p} & 0 \\ 0 & \chi(p)\alpha_{f,p}^{-1} \end{pmatrix} & \text{if } p \nmid D, \\ \alpha_{f,p} & \text{if } p \mid D, \end{cases} \quad \text{and} \quad B_p = \begin{pmatrix} \alpha_{g,p} & 0 \\ 0 & \alpha_{g,p}^{-1} \end{pmatrix}.$$

The L-functions $L(s, f \times g), L(s, f \times g \times \chi)$ are defined by

$$L(s, f \times g) = \prod_{p \nmid D} \det(\mathbf{1}_4 - A_p \otimes B_p \cdot p^{-s})^{-1} \times \prod_{p | D} \det(\mathbf{1}_2 - A_p \otimes B_p \cdot p^{-s})^{-1},$$

$$L(s, f \times g \times \chi) = \prod_{p \nmid D} \det(\mathbf{1}_4 - A_p^{-1} \otimes B_p \cdot p^{-s})^{-1} \times \prod_{p | D} \det(\mathbf{1}_2 - A_p^{-1} \otimes B_p \cdot p^{-s})^{-1}$$

for $\mathrm{Re}(s) \gg 0$. They have holomorphic continuations to the whole s-plane. Note that

$$L(s, f \times g) = \overline{L(\overline{s}, f \times g \times \chi)}$$

by the Ramanujan conjecture. We also define

$$\mathcal{D}(s, f, g) = \sum_{n=1}^{\infty} n^{-s} a_f(n) a_g(n).$$

Then, we have

$$\mathcal{D}(s, f, g) = L(2s - 4\kappa - 1, \chi)^{-1} L(s - 2\kappa - 1/2, f \times g),$$

where $L(s, \chi)$ is the Dirichlet L-function associated to χ. We put $L_\infty(s) = \Gamma_{\mathbb{C}}(s + 1/2)\Gamma_{\mathbb{C}}(s + 2\kappa + 1/2)$ with $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Then they satisfiy the functional equation

$$L_\infty(s) L(s, f \times g) = -D^{1-2s+2\kappa} a_f(D)^{-2} L_\infty(1 - s) L(s, f \times g \times \chi).$$

This equation implies that $a_f(D) L(1/2, f \times g) \in \sqrt{-1}\mathbb{R}$.

The main result is as follows:

Theorem 3. *The identity*

$$L\left(\frac{1}{2}, f \times g\right) = \frac{L(1, \chi)(4\pi)^{2\kappa+1}}{a_f(D)(2\kappa)!} \cdot \frac{1}{h_K} \sum_{[\mathfrak{c}] \in Cl_K} \frac{\langle F_{\mathfrak{c}}|_{\mathfrak{H} \times \mathfrak{H}}, g \times g_C \rangle}{\langle g_C, g_C \rangle}$$

holds.