符号$(1, n)$を持つ完全格子について

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早稲田整数論セミナー

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1 Introduction

Several years ago, I computed 0-cells of the Siegel-Gottschling fundamental domain of degree 2, I noticed not all of them are canonical, but I believe some should be canonial in some sense. T. Watanabe told me about Reshkov domain. The domain is the theoretical support of Voronoi algorithm which solves the sphere packing problem. He generalized these theory to the isotropic reductive algebraic group framework. The symplectic case is included in the theory, and the Siegel's fundamental domain is a fundamental domain in Reshkov domain. The vertices of Reshkov domain are called *perfect*, each of them has strong property, which is the choice of ”canonical”. I come to have an interest on perfect points / perfect lattices / perfect forms. The classification of perfect points is still unsolved in many cases.

2 On a framework of classical groups

Let V be an n-dimensional vector space over $\mathbb{R}$. We fix a lattice $L_0 \subset V$. $GL(V)$ is a set of linear transformations on V and $GL(L_0) = \{g \in GL(V) \mid gL_0 = L_0\}$. We fix $B(x, y)$ an isotropic symmetric or symplectic form

$$B : V \times V \to \mathbb{R}$$

on V. By G or sometimes denoted by $GO(B)$ we denote the set of isometries with siminitude

$$G = \{g \in GL(V) \mid B \circ g \times g \sim B\}$$

and the modular group $\Gamma = G \cap GL(L_0)$.
$\mathcal{N}_k(B)$ is the set of isotropic rank k $\mathbb{Z}$-submodules (simply k-lattices) of L_0 i.e.,

$$\mathcal{N}_k(B) = \{L \subset L_0 \mid \mathrm{rank}(L) = k,\ B(x) = 0\ \ (\forall x \in L)\}$$

We define $\mathcal{N}_k^*(B) \subset \mathcal{N}_k(B)$ the set of primitive isotropic k-lattices. Sometimes we use the notation W_L the $\mathbb{R}$-span of L, and the maximal parabolic subgroup P_L which stabilizes W_L.

3 The ratio of $\det L$

For a lattice L, $\det L$ is the covolume of L. For $g \in G$ and $L \in \mathcal{N}_k(B)$, we define

$$h(g, L) = \frac{\det gL}{\det gL_0^{k/n}} = \frac{\mathrm{vol}(gW_L/gL)}{\mathrm{vol}(gV/gL_0)^{k/n}}$$

Exponent of denominator is chosen suitably to let $h(g, L)$ scaling free. It also holds $h(kg, L) = h(g, L)$ for $k \in K$ the maximal compact subgroup of G and $h(g\gamma, L) = h(g, \gamma L)$ for $\gamma \in \Gamma$.

We describe $\det L$ by matrices. Take a basis $M = \{f_1, \ldots, f_k\}$ of k-lattice L. Then M is a generating matrix of L with size $n \times k$. Using Gram matrix (a positive definite symmetric matrix) $Q = {}^t gg$, we have

$$(\det gL)^2 = \det {}^t(gM)(gM) = \det {}^t MQM = \det Q[M]$$

Therefore

$$h(g, \langle f_1, \ldots, f_k \rangle)^2 = c\frac{\det((Q(f_i, f_j))_{i,j=1}^k)}{\det g^{2k/n}}$$

4 packing functions

The packing function is defined as

$$F(g) = \min_{L \in \mathcal{N}_k(B)} h(g, L) = \min_{L \in \mathcal{N}_k^*(B)} h(g, L)$$

Since $h(g\gamma, L) = h(g, \gamma L)$ for $\gamma \in \Gamma$, $F(g)$ is a function on G/Γ, i.e., a function of G-lattices.

we also have for $\mathcal{N}_k^*(B) = \bigcup_{i=1}^{\mathrm{h}} \Gamma L_i$,

$$F(g) = \min_{1 \le i \le \mathrm{h}} \min_{\gamma \in \Gamma} h(g\gamma, L_i)$$

For example, if $G = O(I_{1,n-1})$, $k = 1$, then h is the class number of $O(n-2)$, or the class number of the quadratic form $x_1^2 + x_2^2 + \cdots + x_{n-2}^2$. Note that $\mathrm{h} = 1$ if $n < 10$. If B is alternating, $n = 2k$ and $G = Sp(2k, \mathbb{R})$, then W_L is maximal totally isotropic space in V and $\mathcal{N}_k^*(B) = \Gamma \begin{pmatrix} I_k \\ 0_k \end{pmatrix}$.

5 Finiteness and minimality

We infer to the finiteness.

Proposition. For a constant T, the cardinality

$$\#\{L \in \mathcal{N}_k^*(B) \mid h(g, L) \leq T\}$$

is finite.

In algebraic group setting, this follows from Northcott's theorem.
In classical group case, however, we can check the finiteness case by case.
When $k = 1$ the following says $h(g, L)$ attains minimum.

Lemma. Let $T > 0$. Let Q be a positive definite symmetric matrix. Then there are only finitely many integral vectors v such that $Q[v] \leq T$.

In general, we can use the following inequality

Lemma. For any n by n positive definite symmetric matrices Q which is Minkowski reduced, there is a constant λ_n such that $\det Q \leq \prod_{i=1}^{n} Q_{ii} \leq \lambda_n \det Q$.

6 Shortest vectors and Perfect property

Define

$$S(g) := \{L \in \mathcal{N}_k^*(B) \mid F(g) = h(g, L)\}$$

$S(g)$ is called the shortest k-vectors (or minimal k-vectors) of g. The cardinality $\#S(g)$ is finite and called the kissing number of g. Note that

$$S(akg\gamma) = \gamma^{-1}S(g) \quad \text{for } a \in Z(G),\ k \in K,\ \gamma \in \Gamma.$$

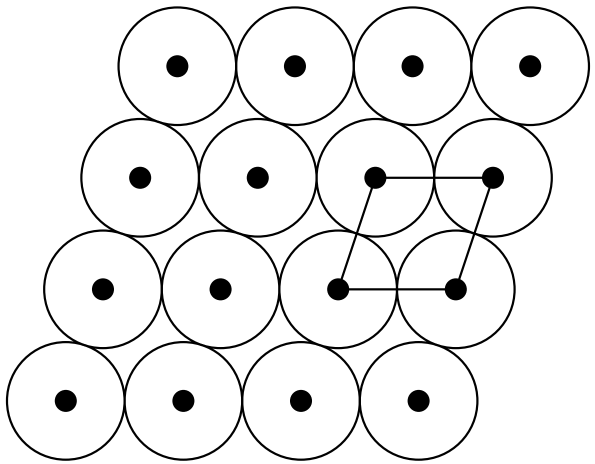
Definition. For $g \in G$ the group element g is called (k-)*perfect* if the kissing number $\#S(g)$ is locally maximal in $g \in Z(G)K\backslash G/\Gamma$.

The definition came from the Voronoi theory; when $G = GL_n(\mathbb{R})$ and $k = 1$, then the lattice gL_0 or quadratic form ${}^{t}gg[x]$ with condition is called the perfect lattice or the perfect form. The primary generatization toward $k > 1$ is found in Rankin [1953], Coulangeon [1996].

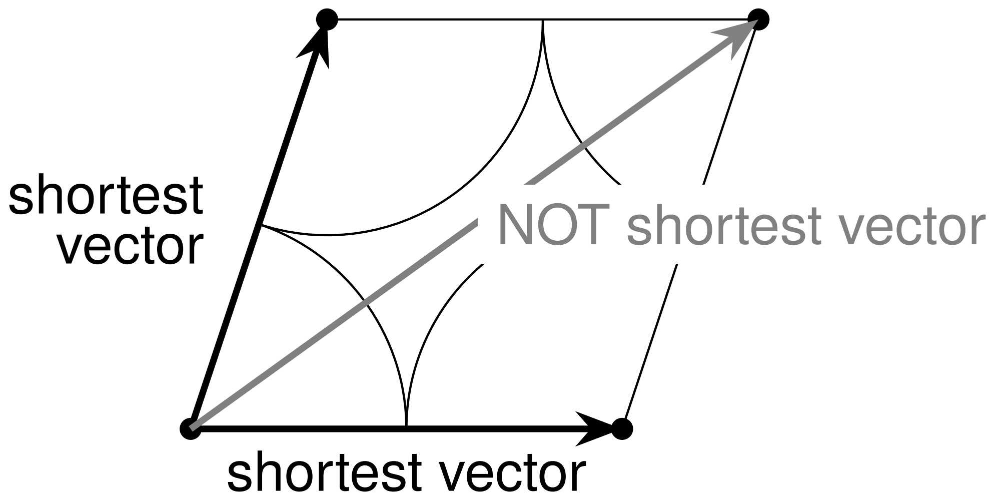
7 Sphere packing problem: $G = GL_n(\mathbb{R})$

Let $V = \mathbb{R}^n$, $L_0 = \mathbb{Z}^n$, $B \equiv 0$ and then $G = GL_n(\mathbb{R})$, $\Gamma = GL_n(\mathbb{Z})$. Consider $k = 1$. Then $\mathcal{N}_1^*(B) = GL_n(\mathbb{Z})e_1$ is the set of rank 1 lattices i.e., primitive integral vectors v modulo ± 1. Then

$$h(g, \langle v \rangle) = \frac{\|gv\|}{\det gL_0^{1/n}} = \frac{Q[v]^{1/2}}{\det(Q)^{1/2n}}$$



$\Rightarrow$



The density of packing is the ratio of a sphere and the fundamental parallelogram, which is proportional to $F(g)$ with some exponent:

$$\text{density of packing} = \frac{\bigcirc}{\square} \sim \frac{(\text{length of shortest vector})^n}{\text{determinant of the lattice}} = F(g)^n$$

The perfect g or the perfect quadratic forms $Q = {}^{t}gg$ is classified upto $n = 8$.

n	2	3	4	5	6	7	8
#	1	1	2	3	7	33	10916
	— before 1900 —				[Barnes 1957]	[Jaquet-Chiffelle 1993]	[Schürmann 2006]

8 Case G rank 1 orthogonal group

We set $V = \mathbb{R}^n$, $L_0 = \mathbb{Z}^n$, $B[x] = I_{1,n-1}[x] = x_1^2 - x_2^2 - \cdots - x_n^2$, $k = 1$. Choose for later use

$$G = GO^+(1,n) = \{g \in GL_n(\mathbb{R}) \mid {}^{\mathrm{t}}g I_{1,n-1} g = \lambda(g) I_{1,n-1}, \quad g_{11} > 0\}$$

which is a rank 1 reductive Lie group. Let $\Omega_n = \{v \in \mathbb{R}^n \mid B[v] > 0, \quad v_1 > 0\}$, the so-called Lorentz cone. Then G acts by the left matrix multiplication on Ω_n. (For consistency, we rewrite the action g on V by $g^{-1}v$.) Let K be a stabilizer of $e_1 \in \Omega_n$. Then $G/K \simeq \Omega_n$.

We can take

$$\mathcal{N}_1^*(B) = \{w \in \mathbb{Z}^n \mid B(w) = 0, \ \gcd(w) = 1, \ w_1 > 0\}$$

Then for $w \in \mathcal{N}_1^*(B)$,

$$h(g,w) = \frac{\|g^{-1}w\|}{|\det g|^{1/n}}, \qquad F(g) = \min_{w \in \mathcal{N}_1^*(B)} h(g,w)$$

9 Passage to Ω_n

Let $g(v) \in G$ is such that $g(v) \cdot e_1 = v$. Import $h(\cdot)$, $F(\cdot)$ on Ω_n by $h(v, w) := h(g(v), w)$, $F(v) := \min_w h(g(v), w)$, and $S(v) := S(g(v))$. Since

$$\det(g(v)) = B(v)^{n/2}, \quad \text{and} \quad g(v)\,{}^{t}g(v) = 2v\,{}^{t}v - B(v)I_{1,n-1}$$

one has $(v \mid w)^2 = \|{}^{t}gv\|^2$ and by taking an inverse of g

$$h(v, w) = \frac{B(v, w)}{(B(v)/2)^{1/2}} = \frac{(v \mid I_{1,n-1}w)}{(B(v)/2)^{1/2}},$$

The shortest vectors of v is then

$$S(v) = \{w \in \mathcal{N}_1^*(B) \mid h(v, w) = F(v)\}$$

10 Reshkov domain

Write $\overline{w} = I_{1,n-1}w$. Choose $\overline{w} \in S(v)$ so that $h(v,w) = F(v)$.
By scaling we can take the representatives of $\Omega_n/Z(G)$ so that $< w, v >= 1$ for $\overline{w} \in S(v)$. Put the Reshkov domain $\mathscr{R}_n$ as

$$\mathscr{R}_n = \{v \in \Omega_n \mid (w \mid v) \geq 1 \ \ \forall w \in \mathscr{N}_1^*(B)\} = \bigcap_{w \in \mathscr{N}_1^*(B)} \{v \in \Omega_n \mid (w \mid v) \geq 1\}$$

Any $v \in \Omega_n/Z(G)$ has a representative on $\partial\mathscr{R}_n$. Since $\mathscr{R}_n$ is the intersection of half spaces, it is convex locally finite polyhedral, since $S(v)$ is a finite set. The point v is perfect if the shortest vectors $S(v)$ is locally maximal. Therefore

Proposition. Perfect v is a vertex(0-cell) of $\mathscr{R}_n$

We then can also check v is perfect iff $\operatorname{rank} S(v) = n$. The 0-cells and 1-cells of $\partial\mathscr{R}_n$ becomes a graph, so a kind of the Voronoi algorithm can be applied to detect all vertices.

11 Results for low dimensional perfect points

The number of Γ-equivalent classes of perfect points on $\mathscr{R}_n$ is obtained completely on $n \leq 12$, partially on $n \geq 13$

n	# of classes	kissing number $\#S(v)$ for perfect points v
3	1	4
4	1	6
5	2	8, 5
6	2	10, 10
7	2	12, 27
8	2	14, 126
9	2	16, 2160
10	4	18, 10, 10, 130
11	9	20, 20, 12, 138, 31, 15, 11, 11, 19
12	9	22, 66, 14, 150, 37, 25, 26, 47, 182
13	≥ 10	24, 232, 16, 166, 47, 45, 61, 167, 7944, 715
14	≥ 19	26, 26, 256, 21, 29, 47, 90, 66, 51, 186, 18, 186, 62
15	≥ 24	28, 196, 280, 99, 60, 55, 62, 77, 110, 226, 55, 28, 27, 71, 55, 194, 41, 53, 31, 19, 23, 20, 210, 83
16	≥ 24	30, 1380, 304, 132, 120, 145, 183, 232, 315, 715, 280, 184, 366, 76, 59, 206, 56, 59, 61, 31, 29, 22, 238, 111

12 Perfect lorentz lattices

For notation $s_n = [1, 0^{n-1}]$, $p_n = [\frac{3}{2}, (\frac{1}{2})^{n-1}]$, $q_n = [\frac{4}{3}, (\frac{1}{3})^{n-1}]$, $r_n = [\frac{7}{5}, (\frac{2}{5})^{n-1}]$, $y_n = [2, 1, (\frac{1}{2})^{n-2}]$, $w_n^{(k)} = [\frac{13}{8}, (\frac{5}{8})^k, (\frac{3}{8})^{n-1-k}]$.

n	#	*lattices*	n	#	*lattices*	n	#	*lattices*	n	#	*lattices*
3	1	s_3	4	1	s_4	5	2	s_5, p_5	6	2	s_6, p_6
7	2	s_7, p_7	8	2	s_8, p_8	9	2	s_9, p_9	10	4	$s_{10}, q_{10}, [r_9, 0], [p_8, 0^2]$

11	9	$s_{11}, q_{11}, [r_9, 0^2], [p_8, 0^3], [p_7, (\frac{1}{6})^4], [p_6, (\frac{1}{4})^5], [p_5, (\frac{3}{10})^6], w_{11}, y_{11},$
12	9	$s_{12}, q_{12}, [r_9, 0^3], [p_8, 0^4], [p_7, (\frac{1}{6})^5], [p_6, (\frac{1}{4})^6], [p_5, (\frac{3}{10})^7], w_{12}^{(1)}, y_{12},$
13	≥10	$s_{13}, q_{13}, [r_9, 0^4], [p_8, 0^5], [p_7, (\frac{1}{6})^6], [p_6, (\frac{1}{4})^7], [p_5, (\frac{3}{10})^8], w_{13}^{(2)}, y^{13}, [\frac{11}{6}, (\frac{1}{2})^{12}], \cdots\cdots$
14	≥19	$s_{14}, [q_{13}, 0], [r_9, 0^5], [p_8, 0^6], [p_7, (\frac{1}{6})^7], [p_6, (\frac{1}{4})^8], [p_5, \frac{1}{10}, (\frac{3}{10})^8], [w_{12}^{(2)}, \frac{1}{4}, 0], [y_{12}, 0^2],$
		$[\frac{13}{10}, (\frac{3}{10})^{13}], [\frac{27}{20}, (\frac{1}{4})^5, (\frac{7}{20})^8], [\frac{11}{8}, (\frac{1}{4})^6, (\frac{3}{8})^7], [\frac{17}{12}, (\frac{1}{4})^7, (\frac{5}{12})^6], \cdots\cdots$
15	≥24	$s_{15}, [q_{13}, 0^2], [q_{12}, (\frac{1}{6})^3], [q_{11}, (\frac{2}{9})^4], [q_{10}, (\frac{1}{4})^5], [p_8, 0^7], [p_7, (\frac{1}{6})^8], [p_6, (\frac{1}{4})^9], [p_5, (\frac{5}{18})^{10}], [y_{12}, 0^3],$
		$[p_5, (\frac{3}{10})^8, (\frac{1}{10})^2], [p_5, (\frac{3}{10})^7, (\frac{1}{5})^3], [p_5, (\frac{3}{10})^6, (\frac{7}{30})^4], [w_{12}^{(2)}, \frac{1}{4}, 0^2], [w_{12}^{(2)}, (\frac{1}{8})^3], [w_{11}^{(2)}, (\frac{5}{24})^4], [y_{11}, (\frac{1}{6})^4],$
		$[r_9, 0^6], [\frac{13}{10}, (\frac{3}{10})^{14}], [\frac{27}{20}, (\frac{7}{20})^8, (\frac{1}{4})^6], [\frac{11}{8}, (\frac{3}{8})^7, (\frac{1}{4})^7], [\frac{17}{12}, (\frac{5}{12})^6, (\frac{1}{4})^8], [\frac{5}{3}, (\frac{2}{3})^2, (\frac{1}{3})^{12}], [\frac{5}{2}, \frac{3}{2}, (\frac{1}{2})^{13}]$
16	≥24	$s_{16}, [q_{13}, 0^3], [q_{12}, (\frac{1}{6})^4], [q_{11}, (\frac{2}{9})^5], [q_{10}, (\frac{1}{4})^6], [p_8, 0^8], [p_7, (\frac{1}{6})^9], [p_6, (\frac{1}{4})^{10}], [p_5, (\frac{5}{18})^{11}], [y_{12}, 0^4],$
		$[p_5, (\frac{3}{10})^6, (\frac{7}{30})^5], [p_5, (\frac{3}{10})^7, (\frac{1}{5})^4], [p_5, (\frac{3}{10})^8, (\frac{1}{10})^3], [w_{12}^{(2)}, \frac{1}{4}, 0^3], [w_{12}^{(2)}, (\frac{1}{8})^4], [w_{11}^{(2)}, (\frac{5}{24})^5], [y_{11}, (\frac{1}{6})^5],$
		$[\frac{13}{10}, (\frac{3}{10})^{15}], [\frac{27}{20}, (\frac{7}{20})^8, (\frac{1}{4})^7], [\frac{11}{8}, (\frac{3}{8})^7, (\frac{1}{4})^8], [\frac{17}{12}, (\frac{5}{12})^6, (\frac{1}{4})^9], [\frac{5}{3}, (\frac{2}{3})^2, (\frac{1}{3})^{13}], [\frac{5}{2}, \frac{3}{2}, (\frac{1}{2})^{14}], [\frac{7}{5}, (\frac{2}{5})^8, 0^7],$

13 Remarks

- When $n = 17$, at least 77 inequivalent perfect points exist.
- s_1 is perfect for all dimension. Its kissing number is $\#S(s_1) = 2(n-1)$. Corresponding Lorentz lattice is $g(s_1)^{-1}L_0 = \mathbb{Z}^n$.
- In $n = 8$, the other perfect lattice $g(p_8)^{-1}L_0$ is of type E_8.
- C. Bavard [2005] computed the extreme Lorentzian lattice completely for $n \leq 12$. Our results match his list.
- We remark $v = [\frac{11}{6}, (\frac{1}{2})^{12}]$ has the property that $[1, 0^{n-2}, -1] \notin \Gamma \cdot S(v)$. In $n = 14$, there are at least 6 inequivalent points with such property.

14 Voronoi algorithm

Voronoi algorithm is a graph traversing algorithm starting from one vertex to adjacent ones.
Two important algorithm are included:

- DD algorithm Computes V(ertex) representation from H(yperplane)-representation of polyhedral convex cone.
- SV algorithm Computes integral vectors satisfying $Q[v] \leq T$ for positive definite Q.

The SV algorithm can also be used for detection of Γ-equivalence.

15 Symplectic case degree 2

Let $n = 4$. Let $V = \mathbb{R}^4 \subset \mathbb{Z}^4 = L_0$ and $k = 2$. We choose $B(x, y) = x_1 y_3 + x_2 y_4 - x_3 y_1 - x_4 y_2$ and $G = Sp(2, \mathbb{R})$ of matrix size 4 and $\Gamma = Sp(2, \mathbb{Z})$. The isotropic 2-lattices can be obtained $\mathcal{N}_2^*(B) = \Gamma \begin{pmatrix} e_1 & e_2 \end{pmatrix}$, which is parametrized by coprime symmetric pairs

$$\mathcal{N}_2^*(B) = GL_2(\mathbb{Z}) \backslash \{(A, C) \in M(4, 2, \mathbb{Z}) \mid \begin{pmatrix} A & * \\ C & * \end{pmatrix} \in \Gamma\}$$

The stabilizer of $\Gamma \begin{pmatrix} e_1 & e_2 \end{pmatrix}$ is the Siegel parabolic subgroup $P = \{ \begin{pmatrix} A & B \\ 0 & {}^{\mathrm{t}}A^{-1} \end{pmatrix} \}$.

G acts on the symmetric space $\mathbb{H}_2 = \{Z = X + iY \mid {}^{\mathrm{t}}Z = Z,\ Y \text{ is positive definite}\}$ by linear fractional transformation. K is then the stabilizer of $Z_0 = \sqrt{-1} I_2$. For $Z = X + iY \in \mathbb{H}_2$, we can take $g(Z) \in G$ so that $g(Z) \cdot Z_0 = Z$ and

$$Q = {}^{\mathrm{t}}g(Z)^{-1} g(Z)^{-1} = \begin{pmatrix} I_2 & 0 \\ -X & I_2 \end{pmatrix} \begin{pmatrix} Y^{-1} & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} I_2 & -X \\ 0 & I_2 \end{pmatrix}$$

Then for block notation $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$,

$$h(Z) := h(g(Z), \langle e_1, e_2 \rangle) = \frac{1}{\det g(Z)^{2/4}} \det Q[e_1\ e_2] = \det Y^{-1} = \det \mathrm{Im}(Z)^{-1}$$

It is well known that

$$\mathrm{Im}(\gamma Z) = {}^{t}\overline{(CZ + D)}^{-1} \, \mathrm{Im}\, Z (CZ + D)^{-1}$$

Therefore $h(\gamma Z) = |\det(CZ + D)|^2 h(Z)$, and so $F(Z) = \min_{\gamma \in \Gamma} |\det(CZ + D)|^2 h(Z)$. By above consideration we can assume $(I_2\ \ O_2) \in S(Z)$, which means Z satisfy $h(Z) = F(Z)$. Consider the reshkov domain $\mathscr{R}_2$ as the set of points $F(Z) = h(Z)$. Namely

$$\begin{aligned}\mathscr{R}_2 &= \{Z \in \mathbb{H}_2 \mid h(Z) = F(Z)\} = \{Z \in \mathbb{H}_2 \mid |\det(CZ + D)|^2 \geq 1 \quad \forall \gamma \in \Gamma\} \\ &= \bigcap_{\gamma \in \Gamma} \{Z \in \mathbb{H}_2 \mid |\det(CZ + D)|^2 \geq 1\}\end{aligned}$$

The shortest 2-vectors of Z is
$S(Z) := S(g(Z)) = GL_2(\mathbb{Z}) \backslash \{(C, D)$ coprime symmetric pair | $h(\gamma Z) = h(Z)\}$.

16 Siegel's fundamental domain

Siegel's fundamental domain $\mathscr{F}_2$ is given by

$$\mathscr{F}_2 = \left\{ Z \in \mathbb{H}_2 \mid \begin{array}{ll} (1) & |X_{ij}| \leq 1/2 \\ (2) & Y \text{ is Minkowski reduced} \\ (3) & |\det(CZ+D)|^2 \geq 1 \ \ \forall \gamma \in \Gamma \end{array} \right\}$$

Theorem. (Gottschling) There are 19 (C, D)'s in (3) which accomplish $\mathscr{F}_2$.

There is a refinement:

Theorem. There are additional 6 (C, D)'s such that

$$\{ Z \in \mathscr{F}_2 \mid |\det(CZ+D)|^2 = 1 \} \neq \emptyset.$$

Z is perfect if $\#S(Z)$ is locally maximal. If we think $\mathscr{F} \subset \mathbb{R}^6$, then $|\det(CZ+D)|^2 = 1$ is a polynomial equation. Note that if Z is perfect, then its representative of Γ-orbit in $\mathscr{F}_2$ is on $\partial\mathscr{F}_2$. If Z is a 0-dimensional solution, it means Z is perfect in the above sense.

17 Result

Assume $\#S(Z) > 6$. There are 3 types of perfect Z in $\mathscr{F}_2$:

symplectic lattice	perfect point	kissing #
$A_2 \times A_2$	$Z = \mathrm{diag}(\frac{1+\sqrt{3}i}{2}, \frac{1+\sqrt{3}i}{2})$	$\#S(Z) = 9$
D_4	$Z = \begin{pmatrix} \frac{1+2\sqrt{2}i}{3} & \frac{-1+\sqrt{3}i}{2} \\ \frac{-1+\sqrt{3}i}{2} & \frac{1+2\sqrt{2}i}{3} \end{pmatrix}$	$\#S(Z) = 8$
e_{18}	$Z = \begin{pmatrix} \frac{1+\sqrt{3}i}{2} & \frac{1+\omega i}{2} \\ \frac{1+\omega i}{2} & \frac{1+\sqrt{3}i}{2} \end{pmatrix}$	$\#S(Z) = 7$

where $\omega^4 - 2\omega^2 - 8\sqrt{3}\omega + 5 = 0$