A proof of Ibukiyama's conjectures on Siegel modular forms of half-integral weight and of degree 2

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Introduction

Theorem (Ibukiyama's conjectures)

Let $k \geq 3$, and $j \geq 0$ even. Then there exists an injective linear map

$$\mathcal{L} : S_{2k-4}(\mathrm{SL}_2(\mathbb{Z})) \otimes S_{2k+2j-2}(\mathrm{SL}_2(\mathbb{Z})) \hookrightarrow S^+_{k-\frac{1}{2},j}(\Gamma_0(4))$$
$$f \otimes g \mapsto \mathcal{L}(f \otimes g)$$

such that if f and g are Hecke eigenforms, then so is $\mathcal{L}(f \otimes g)$, and they satisfy

$$L(s, \mathcal{L}(f \otimes g)) = L(s-j-1, f)L(s, g).$$

Moreover, there exist linear isomorphisms

$$(\mathrm{Im}\,\mathcal{L})^{\perp} \cong S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z})) \cong S^+_{k-\frac{1}{2},j}(\Gamma_0(4), \left(\frac{-1}{\cdot}\right))$$

which preserve the L-functions.

Today's main part

For any integer $k \geq 3$ and even integer $j \geq 0$, there exists an isomorphism

$$\rho : S^+_{k-\frac{1}{2},j}(\Gamma_0(4), \left(\frac{-1}{\cdot}\right)) \xrightarrow{\simeq} S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z}))$$

such that if $F \in S^+_{k-\frac{1}{2},j}(\Gamma_0(4), \left(\frac{-1}{\cdot}\right))$ is a Hecke eigenform, then so is $\rho(F) \in S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z}))$, and they satisfy

$$L(s, F) = L(s, \rho(F), \mathrm{spin}).$$

Tools

- Arthur's multiplicity formula for SO_5
- Gan-Ichino's multiplicity formula for Mp_4
- Jacobi forms and Jacobi groups

Sketch

$$\begin{array}{ccccc}
S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z})) & \rightsquigarrow & L^2_{\mathrm{disc}}(\mathrm{SO}_5(\mathbb{Q})\backslash \mathrm{SO}_5(\mathbb{A}_{\mathbb{Q}})) & \longleftrightarrow & (A\text{-parameters}) \\
\vdots & & & & \| \\
S^+_{k-\frac{1}{2},j}(\Gamma_0(4), \left(\frac{-1}{\cdot}\right)) & & L^2_{\mathrm{disc}}(\mathrm{Sp}_4(\mathbb{Q})\backslash \mathrm{Mp}_4(\mathbb{A}_{\mathbb{Q}})) & \longleftrightarrow & (A\text{-parameters}) \\
\downarrow \wr & & \updownarrow & & \\
\{ \text{ Jacobi cusp form } \} & \rightsquigarrow & L^2_{\mathrm{disc}}(\mathrm{Sp}^J_4(\mathbb{Q})\backslash \mathrm{Sp}^J_4(\mathbb{A}_{\mathbb{Q}})) & &
\end{array}$$

Remarks:

- We have an accidental isomorphism $\mathrm{PGSp}_4 \cong \mathrm{SO}_5$;
- The A-parameters for Mp_4 are the same as those for SO_5.

Integral weight Siegel cusp forms 1

- $\mathfrak{H}_n = \{\, Z = X + iY \in \mathrm{M}_n(\mathbb{C}) \mid X = {}^{\mathrm{t}}X, \quad Y = {}^{\mathrm{t}}Y > 0 \,\}$
- $\mathrm{GSp}_{2n} = \{\, g \in \mathrm{GL}_{2n} \mid {}^{\mathrm{t}}gJ_ng = \nu(g)J_n,\ \nu(g) \in \mathrm{GL}_1 \,\}$, $\mathrm{Sp}_{2n} = \ker(\nu)$
- $\mathrm{GSp}^+_{2n}(\mathbb{R}) = \{\, g \in \mathrm{GSp}_{2n}(\mathbb{R}) \mid \nu(g) > 0 \,\} \curvearrowright \mathfrak{H}_n$, $J(g, Z) = CZ + D$
  - $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $J_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$
- (Sym_j, V_j) : the symmetric tensor rep. of degree j of $\mathrm{GL}_2(\mathbb{C})$
- $[f|_{k,j}g]\,(Z) = \nu(g)^{k+\frac{j}{2}} \det(J(g,Z))^{-k}\,\mathrm{Sym}_j(J(g,Z))^{-1} f(gZ)$
  - $f : \mathfrak{H}_2 \to V_j$, $g \in \mathrm{GSp}^+_4(\mathbb{R})$
  - (k, j) means the weight $\det^k \mathrm{Sym}_j$.

$S_{k,j}(\mathrm{Sp}_4(\mathbb{Z})) = \{\, f : \mathfrak{H}_2 \to V_j \mid f|_{k,j}\gamma = f,\ \forall\gamma \in \mathrm{Sp}_4(\mathbb{Z}), \text{holom., cusp} \,\}$

Integral weight Siegel cusp forms 2

$f \in S_{k,j}(\mathrm{Sp}_4(\mathbb{Z}))$, $m \in \mathbb{Z}_{>0}$

- $X(m) = \{\, x \in \mathrm{Mat}_4(\mathbb{Z}) \cap \mathrm{GSp}_4^+(\mathbb{R}) \mid \nu(x) = m \,\}$
- Hecke operators

$$f|_{k,j}T(m) = m^{k+\frac{j}{2}-3} \sum_{g \in \mathrm{Sp}_4(\mathbb{Z})\backslash X(m)} f|_{k,j}g$$

f : Hecke eigenform

$\rightsquigarrow$ the spinor L-function

$$L(s,f,\mathrm{spin}) = \prod_p L(s,f,\mathrm{spin})_p$$

Half-integral weight Siegel cusp forms 1

- $\widetilde{\mathrm{GSp}_4^+}(\mathbb{R})$: the 4-fold covering group of $\mathrm{GSp}_4^+(\mathbb{R})$:
  - $\left\{ (g,\phi(Z)) \in \mathrm{GSp}_4^+(\mathbb{R}) \times \mathrm{Hol}(\mathfrak{H}_2,\mathbb{C}) \;\middle|\; \phi(Z)^4 = \frac{\det J(g,Z)^2}{\det g} \right\}$
  - $(g_1,\phi_1(Z))(g_2,\phi_2(Z)) = (g_1g_2, \phi_1(g_2Z)\phi_2(Z))$
- $\theta(Z) = \sum_{x\in\mathbb{Z}^2} \mathrm{e}({}^t xZx), \quad Z \in \mathfrak{H}_2,\ \mathrm{e}(z) = \exp(2\pi iz)$
  - an embedding $\Gamma_0(4) \hookrightarrow \widetilde{\mathrm{GSp}_4^+}(\mathbb{R})$, $\gamma \mapsto \left(\gamma, \frac{\theta(\gamma Z)}{\theta(Z)}\right)$
- $\left(\frac{-1}{\gamma}\right) = \left(\frac{-1}{\det D}\right)$, $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(4)$
- $\left[F|_{k-\frac{1}{2},j}(g,\phi(Z))\right](Z) = \nu(g)^{\frac{j}{2}}\phi(Z)^{-2k+1}\,\mathrm{Sym}_j(J(g,Z))^{-1}F(gZ)$
  - ◀ $F : \mathfrak{H}_2 \to V_j$, $(g,\phi(Z)) \in \widetilde{\mathrm{GSp}_4^+}(\mathbb{R})$

$$S_{k-\frac{1}{2},j}(\Gamma_0(4),\left(\frac{-1}{\cdot}\right)) =$$
$$\left\{ F : \mathfrak{H}_2 \to V_j \;\middle|\; F|_{k-\frac{1}{2},j}\gamma = \left(\frac{-1}{\gamma}\right)F,\ \forall\gamma\in\Gamma_0(4), \quad \text{holom., cusp} \right\}$$

Half-integral weight Siegel cusp forms 2

$F \in S_{k-\frac{1}{2},j}(\Gamma_0(4),(\frac{-1}{\cdot}))$

- Fourier series expansion $F(Z) = \sum_T A(T)\mathrm{e}(\mathrm{tr}(TZ))$
    - T runs over positive definite half-integral symmetric matrices of degree 2.

$S^+_{k-\frac{1}{2},j}(\Gamma_0(4),(\frac{-1}{\cdot}))$: the set of $F \in S_{k-\frac{1}{2},j}(\Gamma_0(4),(\frac{-1}{\cdot}))$ such that $A(T) = 0$ unless $T \equiv (-1)^k r\,{}^{\mathrm{t}}r \mod 4$ for some $r \in \mathbb{Z}^2$

Half-integral weight Siegel cusp forms 3

Hecke operators at odd prime $p \neq 2$

- $K_s(p^2) = \begin{pmatrix} 1_{2-s} & & & \\ & p1_s & & \\ & & p^2 1_{2-s} & \\ & & & p1_s \end{pmatrix}, \ s = 0, 1, 2.$
- $\Gamma_0(4)(K_s(p^2), p^{1-\frac{s}{2}})\Gamma_0(4) = \bigsqcup_t \Gamma_0(4)\widetilde{g}_{s,t}$ in $\widetilde{\mathrm{GSp}_4^+(\mathbb{R})}$
- $F|_{k-\frac{1}{2},j} T_s(p) = \sum_t \left(\frac{-1}{g_{s,t}}\right) F|_{k-\frac{1}{2},j}\widetilde{g}_{s,t}, \quad \widetilde{g}_{s,t} = (g_{s,t}, *)$

$\rightsquigarrow$ the Euler p-factor $L(s, F)_p$

2 is a bad prime for $\Gamma_0(4)$.
We use Jacobi forms to define $T_s(2)$ and $L(s, F)_2$.

Holomorphic and skew-holomorphic Jacobi cusp forms 1

- the Jacobi group $\mathrm{Sp}_4^J = \mathrm{Sp}_4 \ltimes \mathcal{H}_2 \subset \mathrm{Sp}_6$
  - an embedding $\mathrm{Sp}_4 \hookrightarrow \mathrm{Sp}_6$, $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} 1 & & & \\ & A & & B \\ & & 1 & \\ & C & & D \end{pmatrix}$
  - the Heisenberg group

$$\mathcal{H}_2 = \left\{ ([\lambda,\mu],\kappa) = \begin{pmatrix} 1 & {}^t\lambda & \kappa & {}^t\mu \\ & 1_2 & \mu & \\ & & 1 & \\ & & -\lambda & 1_2 \end{pmatrix} \in \mathrm{Sp}_6 \right\}$$

$$J_{(k,j)}^{\mathrm{hol,cusp}} = \left\{ F : \mathfrak{H}_2 \times \mathbb{C}^2 \to V_j \;\middle|\; F|_{(k,j)}^{\mathrm{hol}}\gamma = F, \forall\gamma \in \mathrm{Sp}_4^J(\mathbb{Z}), \quad \text{holom., cusp} \right\}$$

$$J_{(k,j)}^{\mathrm{skew,cusp}} = \left\{ F : \mathfrak{H}_2 \times \mathbb{C}^2 \to V_j \;\middle|\; F|_{(k,j)}^{\mathrm{skew}}\gamma = F, \forall\gamma \in \mathrm{Sp}_4^J(\mathbb{Z}), \quad \text{skew-holom., cusp} \right\}$$

Holomorphic and skew-holomorphic Jacobi cusp forms 2

Slash operators

- For $([\lambda,\mu],\kappa) \in \mathcal{H}_2(\mathbb{R})$ and $F : \mathfrak{H}_2 \times \mathbb{C}^2 \to V_j$,

$$\left[F|^{\mathrm{hol}}_{(k,j)}([\lambda,\mu],\kappa)\right](Z,w) = \left[F|^{\mathrm{skew}}_{(k,j)}([\lambda,\mu],\kappa)\right](Z,w)$$
$$= \mathrm{e}({}^{\mathrm{t}}\lambda Z\lambda + 2\,{}^{\mathrm{t}}\lambda w + {}^{\mathrm{t}}\lambda\mu + \kappa)F(Z, w + Z\lambda + \mu)$$

- For $g \in \mathrm{GSp}_4^+(\mathbb{R})$ and $F : \mathfrak{H}_2 \times \mathbb{C}^2 \to V_j$,

$$\left[F|^{\mathrm{hol}}_{(k,j)}g\right](Z,w) = \nu(g)^{k+\frac{j}{2}}\mathrm{e}(-\,{}^{\mathrm{t}}wJ(g,Z)^{-1}Cw)$$
$$\times \det(J(g,Z))^{-k}\,\mathrm{Sym}_j(J(g,Z))^{-1}F(gZ, \nu(g)^{\frac{1}{2}}\,{}^{\mathrm{t}}J(g,Z)^{-1}w)$$
$$\left[F|^{\mathrm{skew}}_{(k,j)}g\right](Z,w) = \nu(g)^{k+\frac{j}{2}}\mathrm{e}(-\,{}^{\mathrm{t}}wJ(g,Z)^{-1}Cw)$$
$$\times \frac{|\det J(g,Z)|}{\det J(g,Z)}\overline{\det J(g,Z)^{-k}\,\mathrm{Sym}_j(J(g,Z))^{-1}}F(gZ, \nu(g)^{\frac{1}{2}}\,{}^{\mathrm{t}}J(g,Z)^{-1}w)$$

Holomorphic and skew-holomorphic Jacobi cusp forms 3

$F : \mathfrak{H}_2 \times \mathbb{C}^2 \to V_j$ is skew-holomorphic.

$\overset{\text{def}}{\Leftrightarrow}$ $F(Z, w)$ is holomorphic in $w \in \mathbb{C}^2$ and real analytic in the real part X and imaginary part Y of $Z \in \mathfrak{H}_2$.

Cusp condition

- $F \in J^{\mathrm{hol,cusp}}_{(k,j)}$ has a Fourier expansion

$$F(Z, w) = \sum_{\substack{(N,r) \in L_2^* \times \mathbb{Z}^2 \\ 4N - r\,{}^{\mathrm{t}}r > 0}} A(N, r) \mathrm{e}(\mathrm{tr}(NZ) + {}^{\mathrm{t}}rw)$$

- $F \in J^{\mathrm{skew,cusp}}_{(k,j)}$ has a Fourier expansion

$$F(Z, w) = \sum_{\substack{(N,r) \in L_2^* \times \mathbb{Z}^2 \\ 4N - r\,{}^{\mathrm{t}}r < 0}} A(N, r) \mathrm{e}(\mathrm{tr}(NZ - \frac{1}{2} i(4N - r\,{}^{\mathrm{t}}r)Y)) \mathrm{e}({}^{\mathrm{t}}rw)$$

◀ L_2^* : the set of all half-integral symmetric matrices

Holomorphic and skew-holomorphic Jacobi cusp forms 4

Hecke operators at any prime p including 2

- $K_s(p^2) = \begin{pmatrix} 1_{2-s} & & & \\ & p1_s & & \\ & & p^2 1_{2-s} & \\ & & & p1_s \end{pmatrix}$, $(s = 0, 1, 2)$: as before
- $\mathrm{Sp}_4(\mathbb{Z}) K_s(p^2)\, \mathrm{Sp}_4(\mathbb{Z}) = \bigsqcup_t \mathrm{Sp}_4(\mathbb{Z}) g_{s,t}$
- $F|^{\star}_{(k,j)} T^J_s(p) = \sum_{\lambda,\mu \in (\mathbb{Z}/p\mathbb{Z})^2} \sum_t F|^{\star}_{(k,j)} g_{s,t}|^{\star}_{(k,j)}([\lambda,\mu], 0)$
  - $F \in J^{\star,\mathrm{cusp}}_{(k,j)}$, $\star = \mathrm{hol}, \mathrm{skew}$

Half-integral weight Siegel cusp forms 4

Theorem (Ibukiyama et al.)

There exists a linear isomorphism

$$\sigma : J^{\star,\mathrm{cusp}}_{(k,j)} \xrightarrow{\simeq} S^+_{k-\frac{1}{2},j}\left(\Gamma_0(4), \left(\frac{-1}{\cdot}\right)\right), \quad \star = \begin{cases} \mathrm{hol} & k \text{ odd} \\ \mathrm{skew} & k \text{ even} \end{cases}$$

such that

$$F|^{\star}_{(k,j)} T^J_s(p) = p^{3+\frac{s}{2}} \left(\frac{-1}{p}\right)^s \sigma(F)|_{k-\frac{1}{2},j} T_s(p),$$

for any odd prime p.

- ⇝ $T_s(2)$ on $S^+_{k-\frac{1}{2},j}(\Gamma_0(4), \left(\frac{-1}{\cdot}\right))$
- ⇝ Euler 2-factor $L(s,F)_2$ and L-function $L(s,F) = \prod_p L(s,F)_p$

Main theorem (again)

Main theorem (Ibukiyama's conjecture)

For any integer $k \geq 3$ and even integer $j \geq 0$, there exists an isomorphism

$$\rho : S^+_{k-\frac{1}{2},j}(\Gamma_0(4),\left(\frac{-1}{\cdot}\right)) \xrightarrow{\simeq} S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z}))$$

such that if $F \in S^+_{k-\frac{1}{2},j}(\Gamma_0(4),\left(\frac{-1}{\cdot}\right))$ is a Hecke eigenform, then so is $\rho(F) \in S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z}))$, and they satisfy

$$L(s,F) = L(s,\rho(F),\mathrm{spin}).$$

Multiplicity formulae

A-parameters

The notion of elliptic A-parameters for SO_5 and for Mp_4 are the same. An elliptic A-parameter for SO_5 or Mp_4 is a formal direct sum

$$\phi = \bigoplus_i \phi_i \boxtimes S_{d_i},$$

where

- ϕ_i : an irr. self-dual cuspidal automorphic rep. of $\mathrm{GL}_{n_i}(\mathbb{A}_{\mathbb{Q}})$;
- S_d : the irr. d-dimensional rep. of $\mathrm{SL}_2(\mathbb{C})$;
- d_i : odd $\Rightarrow \phi_i$: symplectic;
- d_i : even $\Rightarrow \phi_i$: orthogonal;
- $i \neq j \Rightarrow (\phi_i, d_i) \neq (\phi_j, d_j)$;
- $\sum_i n_i d_i = 4$.

The localization at a place v (of $\mathbb{Q}$)

$$\phi_v = \bigoplus_i \phi_{i,v} \boxtimes S_{d_i} : L_{\mathbb{Q}_v} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_4(\mathbb{C})$$

- ◀ $\phi_{i,v}$ is identified with its L-parameter, via the LLC for GL_n.
- ◀ $L_{\mathbb{Q}_v}$: the Langlands group of $\mathbb{Q}_v$

A-parameters (explicit)

The A-parameter ϕ for SO_5 or Mp_4 is one of the following forms.

(1) $\phi = \chi \boxtimes S_4$

(2) $\phi = \chi \boxtimes S_2 \oplus \chi' \boxtimes S_2$

(3) $\phi = \sigma \boxtimes S_2$

(4) $\phi = \chi \boxtimes S_2 \oplus \mu \boxtimes S_1$

(5) $\phi = \mu \boxtimes S_1 \oplus \mu' \boxtimes S_1$

(6) $\phi = \tau \boxtimes S_1$

- χ, χ' : quadratic characters of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$
- σ : an irr. cuspidal orthogonal autom. rep. of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$
- μ, μ' : irr. cuspidal symplectic autom. rep.s of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$
- τ : an irr. cuspidal symplectic autom. rep. of $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$

Multiplicity formula for SO_5

For an elliptic A-parameter $\phi = \bigoplus_i \phi_i \boxtimes S_{d_i}$,

- The global component group $S_\phi = \bigoplus_i (\mathbb{Z}/2\mathbb{Z}) a_i$
- The local component group $S_{\phi_v} = \pi_0[\mathrm{Cent}(\mathrm{im}(\phi_v), \mathrm{Sp}_4(\mathbb{C}))]$

$\rightsquigarrow$ $S_\phi \to S_{\phi_v}$, and $\Delta : S_\phi \to S_{\phi,\mathbb{A}} = \prod_v S_{\phi_v}$

Multiplicity formula for SO_5 (Arthur)

For the split SO_5, Arthur gave a character $\epsilon_\phi \in \widehat{S}_\phi$ and finite sets $\Pi_{\phi_v}(SO_5) = \left\{ \sigma_{\eta_v} \;\middle|\; \eta_v \in \widehat{S}_{\phi_v} \right\}$ of semisimple representations of $SO_5(\mathbb{Q}_v)$ of finite length indexed by characters of S_{ϕ_v} such that

$$L^2_{\mathrm{disc}}(SO_5(\mathbb{Q})\backslash SO_5(\mathbb{A}_\mathbb{Q})) = \bigoplus_\phi \bigoplus_{\eta \in \widehat{S}_{\phi,\mathbb{A}}} n_\eta \sigma_\eta, \quad \sigma_\eta = \otimes_v \sigma_{\eta_v},$$

where $\eta = \otimes_v \eta_v$ and $n_\eta = \begin{cases} 1, & \text{if } \eta \circ \Delta = \epsilon_\phi, \\ 0, & \text{otherwise.} \end{cases}$

The metaplectic group Mp_4

The metaplectic group Mp_4 is the nontrivial 2-fold covering group of Sp_4. (It is not an algebraic group.)

- $1 \to \{\pm 1\} \to \mathrm{Mp}_4(\mathbb{Q}_v) \to \mathrm{Sp}_4(\mathbb{Q}_v) \to 1$
  - a canonical splitting $\mathrm{Sp}_4(\mathbb{Z}_p) \hookrightarrow \mathrm{Mp}_4(\mathbb{Q}_p)$ for $p \neq 2, \infty$
  - ⇝ the notion of spherical representations for $p \neq 2$
- $1 \to \{\pm 1\} \to \mathrm{Mp}_4(\mathbb{A}_\mathbb{Q}) \to \mathrm{Sp}_4(\mathbb{A}_\mathbb{Q}) \to 1$
  - a canonical splitting $\mathrm{Sp}_4(\mathbb{Q}) \hookrightarrow \mathrm{Mp}_4(\mathbb{A}_\mathbb{Q})$
  - ⇝ the notion of automorphic forms $\varphi : \mathrm{Sp}_4(\mathbb{Q}) \backslash \mathrm{Mp}_4(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$

- A representation of Mp_4 is said to be genuine if it does not factor through the covering group $\mathrm{Mp}_4 \to \mathrm{Sp}_4$.
- $L^2_{\mathrm{disc}}(\mathrm{Mp}_4)$: the genuine discrete spectrum of $L^2(\mathrm{Sp}_4(\mathbb{Q}) \backslash \mathrm{Mp}_4(\mathbb{A}_\mathbb{Q}))$

Multiplicity formula for Mp_4

- Recall S_ϕ, S_{ϕ_v}, and $\Delta : S_\phi \to S_{\phi,\mathbb{A}} = \prod_v S_{\phi_v}$.
- Fix an additive character $\psi : \mathbb{Q}\backslash\mathbb{A}_\mathbb{Q} \to \mathbb{C}^1$ s.t. $\psi_\infty = \mathrm{e}$.

Multiplicity formula for Mp_4 (Gan-Ichino)

Gan-Ichino gave a character $\widetilde{\epsilon}_\phi \in \widehat{S}_\phi$ and finite sets (dependent on ψ_v) $\Pi_{\phi_v,\psi_v}(\mathrm{Mp}_4) = \left\{ \pi_{\eta_v,\psi_v} \,\middle|\, \eta_v \in \widehat{S}_{\phi_v} \right\}$ of semisimple representations of $\mathrm{Mp}_4(\mathbb{Q}_v)$ of finite length indexed by characters of S_{ϕ_v} such that

$$L^2_{\mathrm{disc}}(\mathrm{Mp}_4) = \bigoplus_{\phi} \bigoplus_{\eta\in\widehat{S}_{\phi,\mathbb{A}}} m_\eta \pi_{\eta,\psi}, \quad \pi_{\eta,\psi} = \otimes_v \pi_{\eta_v,\psi_v},$$

where $m_\eta = \begin{cases} 1, & \text{if } \eta\circ\Delta = \widetilde{\epsilon}_\phi, \\ 0, & \text{otherwise.} \end{cases}$

Sketch (again)

$$\begin{array}{ccccc}
S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z})) & \rightsquigarrow & L^2_{\mathrm{disc}}(\mathrm{SO}_5(\mathbb{Q})\backslash \mathrm{SO}_5(\mathbb{A}_{\mathbb{Q}})) & \longleftrightarrow & (A\text{-parameters}) \\
\vdots & & & & \| \\
S^+_{k-\frac{1}{2},j}(\Gamma_0(4),\left(\frac{-1}{\cdot}\right)) & & L^2_{\mathrm{disc}}(\mathrm{Mp}_4) & \longleftrightarrow & (A\text{-parameters}) \\
\downarrow \wr & & \updownarrow & & \\
J^{\star,\mathrm{cusp}}_{(k,j)} & \rightsquigarrow & L^2_{\mathrm{disc}}(\mathrm{Sp}_4^J(\mathbb{Q})\backslash \mathrm{Sp}_4^J(\mathbb{A}_{\mathbb{Q}})) & &
\end{array}$$

$S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z})) \rightsquigarrow L^2_{\mathrm{disc}}(\mathrm{SO}_5(\mathbb{Q})\backslash \mathrm{SO}_5(\mathbb{A}_{\mathbb{Q}}))$

- $f \in S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z}))$: a Hecke eigenform

$\rightsquigarrow$ $\Phi_f(g) = [f|_{j+3,2k-6}g_\infty](i1_2)$, for $g = \gamma g_\infty \kappa \in \mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$

- strong approximation $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}}) = \mathrm{GSp}_4(\mathbb{Q})\,\mathrm{GSp}_4^+(\mathbb{R})\,\mathrm{GSp}_4(\hat{\mathbb{Z}})$
- level $\mathrm{Sp}_4(\mathbb{Z})$

$\rightsquigarrow$ $\varphi_{f,v}(g) = \langle v, \Phi_f(g)\rangle$, for $v \in V^\vee_{2k-6}$

$\rightsquigarrow$ π_f : an irr. cuspidal autom. rep. of $\mathrm{PGSp}_4(\mathbb{A}_{\mathbb{Q}}) \cong \mathrm{SO}_5(\mathbb{A}_{\mathbb{Q}})$

- $\pi_{f,\infty}$: holom. d.s. w/ lowest K-type $(k-3, k+j)$
- $\pi_{f,p}$: spherical (unramified) w/ same Satake parameter as f, for all p

$\rightsquigarrow$ ϕ_f : the A-parameter of π_f (Arthur's multiplicity formula)

$S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z})) \longleftrightarrow$ (A-parameters)

Lemma (essentially by Chenevier-Lannes)

We have $\phi_f = \tau_f \boxtimes S_1$,
$\exists \tau_f$: an irr. symplectic cuspidal autom. rep. of $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$ s.t.
$L(s, f, \mathrm{spin}) = L(s - j - k + \frac{3}{2}, \tau_f)$ and $\tau_{f,\infty} = \mathcal{D}_{k+j-\frac{3}{2}} \oplus \mathcal{D}_{k-\frac{5}{2}}$.
Moreover, $\mathbb{C}f \mapsto \tau_f$ is a bijection between
{1-dim'l Hecke eigenspace in $S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z}))$} and
{irr. symp. cusp. autom. rep. of GL_4 that is unramified at any p and $\mathcal{D}_{k+j-\frac{3}{2}} \oplus \mathcal{D}_{k-\frac{5}{2}}$ at ∞}.

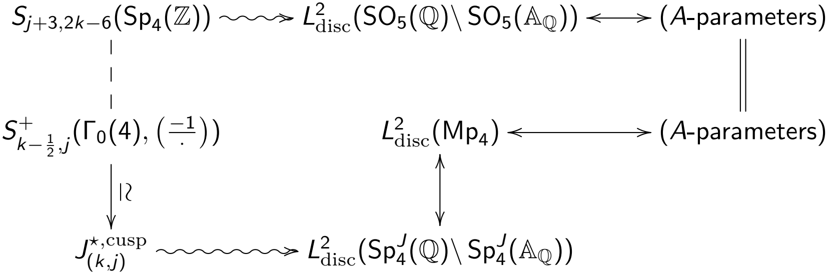
◀ $\mathcal{D}_a = \mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}}(re^{i\theta} \mapsto e^{2ia\theta})$

$\because$) Rule out other possibilities by using the conditions:

- $\pi_{f,\infty}$: holom. d.s. w/ lowest K-type $(k-3, k+j)$
- $\pi_{f,p}$: unramified w/ same Satake parameter as f, for all p
- j : even

For the last assertion, do the converse. □

Sketch (again)



$S^+_{k-\frac{1}{2},j}(\Gamma_0(4),(\frac{-1}{\cdot})) \rightsquigarrow L^2_{\mathrm{disc}}(\mathrm{Sp}^J_4(\mathbb{Q})\backslash \mathrm{Sp}^J_4(\mathbb{A}_\mathbb{Q}))$

- $F \in J^{\star,\mathrm{cusp}}_{(k,j)}$: a Hecke eigenform, $F' = \sigma(F) \in S^+_{k-\frac{1}{2},j}(\Gamma_0(4),(\frac{-1}{\cdot}))$

$\rightsquigarrow$ $\Phi_F(g) = [F|^\star_{k,j} g_\infty](i1_2, 0)$, for $g = \gamma g_\infty \kappa \in \mathrm{Sp}^J_4(\mathbb{A}_\mathbb{Q})$

- strong approximation $\mathrm{Sp}^J_4(\mathbb{A}_\mathbb{Q}) = \mathrm{Sp}^J_4(\mathbb{Q})\,\mathrm{Sp}^J_4(\mathbb{R})\,\mathrm{Sp}^J_4(\hat{\mathbb{Z}})$
- level $\mathrm{Sp}^J_4(\mathbb{Z})$

$\rightsquigarrow$ $\varphi_{F,v}(g) = \langle v, \Phi_F(g)\rangle$, for $v \in V_j^\vee$

$\rightsquigarrow$ π_F : an irr. cuspidal autom. rep. of $\mathrm{Sp}^J_4(\mathbb{A}_\mathbb{Q})$

- $\pi_{F,\infty}$: d.s. w/ a specific lowest K-type
- $\pi_{F,p}$: spherical (unramified) w/ same Satake parameter as F, $\forall p$
- The center of Sp^J_4 is $\mathcal{Z} = \{([0,0],z) \in \mathcal{H}_2\} \cong \mathbb{G}_a$.
  π_F has the central character $\psi : \mathbb{Q}\backslash\mathbb{A}_\mathbb{Q} \to \mathbb{C}^1$ w/ $\psi_\infty = \mathrm{e}$.

- $L^2_{\mathrm{disc}}(\mathrm{Sp}^J_4(\mathbb{Q})\backslash \mathrm{Sp}^J_4(\mathbb{A}_\mathbb{Q}))_\psi$: the max. subspace on which $\mathcal{Z}(\mathbb{A}_\mathbb{Q})$ acts by ψ.

$L^2_{\mathrm{disc}}(\mathrm{Sp}^J_4(\mathbb{Q})\backslash \mathrm{Sp}^J_4(\mathbb{A}_\mathbb{Q}))_\psi \longleftrightarrow L^2_{\mathrm{disc}}(\mathrm{Mp}_4)$ 1

- $\pi_{S,\psi}$: the Schrödinger rep. of $\mathcal{H}_2(\mathbb{A}_\mathbb{Q})$ w/ cent. char. ψ
- ω_ψ : the Weil rep. of $\mathrm{Mp}_4(\mathbb{A}_\mathbb{Q})$ rel. to ψ

$\rightsquigarrow$ $\pi_{SW,\psi}$: the Schrödinger-Weil rep. of $\mathrm{Mp}_4(\mathbb{A}_\mathbb{Q}) \ltimes \mathcal{H}_2(\mathbb{A}_\mathbb{Q})$

- The Stone-von Neumann theorem gives the following fact.

Proposition (essentially by Berndt-Schmidt)

For any irr. subrep. $\pi' \subset L^2_{\mathrm{disc}}(\mathrm{Mp}_4)$,
we have an irr. subrep. $\pi = \pi' \otimes \pi_{SW,\psi} \subset L^2_{\mathrm{disc}}(\mathrm{Sp}^J_4(\mathbb{Q})\backslash \mathrm{Sp}^J_4(\mathbb{A}_\mathbb{Q}))_\psi$.
This is a bijective correspondence.

- Similar bijective correspondences exist over local fields.

$\cdots$ compatible with the global correspondence:

$$\pi' = \bigotimes_v \pi'_v \Rightarrow \pi = \pi' \otimes \pi_{SW,\psi} = \bigotimes_v (\pi'_v \otimes \pi_{SW,\psi_v})$$

$L^2_{\mathrm{disc}}(\mathrm{Sp}_4^J(\mathbb{Q})\backslash \mathrm{Sp}_4^J(\mathbb{A}_\mathbb{Q}))_\psi \longleftrightarrow L^2_{\mathrm{disc}}(\mathrm{Mp}_4)$ 2

- $\pi_F \subset L^2_{\mathrm{disc}}(\mathrm{Sp}_4^J(\mathbb{Q})\backslash \mathrm{Sp}_4^J(\mathbb{A}_\mathbb{Q}))_\psi \rightsquigarrow \pi'_F \subset L^2_{\mathrm{disc}}(\mathrm{Mp}_4)$
  - $\pi'_{F,\infty}$: d.s., w/ lowest K-type $\begin{cases}(k+j-\frac{1}{2},k-\frac{1}{2}) & \star=\mathrm{hol}\\ (-k+\frac{1}{2},-k-j+\frac{1}{2}) & \star=\mathrm{skew}\end{cases}$
  - $\pi'_{F,p}$:The followings can be seen. Note that $\pi_{F,p}=\pi'_{F,p}\otimes\pi_{SW,\psi_p}$

Proposition

$\pi'_{F,p}$ is spherical, and hence unramified for $p\neq 2$.

$\because$) If $p\neq 2$, then π_{SW,ψ_p} is spherical. □

$L^2_{\mathrm{disc}}(\mathrm{Sp}_4^J(\mathbb{Q})\backslash \mathrm{Sp}_4^J(\mathbb{A}_{\mathbb{Q}}))_\psi \longleftrightarrow L^2_{\mathrm{disc}}(\mathrm{Mp}_4)$ 2

Proposition

$\pi'_{F,p}$ is unramified for any prime p including 2.

$\because$) Intertwining operators for $\mathrm{Sp}_4^J(\mathbb{Q}_p)$ and $\mathrm{Mp}_4(\mathbb{Q}_p)$ are compatible with the correspondence $\pi'_{F,p} \mapsto \pi_{F,p} = \pi'_{F,p} \otimes \pi_{SW,\psi_p}$.

$$\begin{array}{ccc}
(\text{a std. module of } \mathrm{Sp}_4^J(\mathbb{Q}_p)) & \longleftrightarrow & (\text{a std. module of } \mathrm{Mp}_4(\mathbb{Q}_p)) \\
\downarrow \text{intertwining op.} & & \downarrow \text{intertwining op.} \\
(\text{an unram. rep. of } \mathrm{Sp}_4^J(\mathbb{Q}_p)) & \longleftrightarrow & (\text{an unram. rep. of } \mathrm{Mp}_4(\mathbb{Q}_p))
\end{array}$$

$\square$

$\rightsquigarrow$ ϕ_F : the A-parameter of π'_F (Gan-Ichino's multiplicity formula)

$S^+_{k-\frac{1}{2},j}(\Gamma_0(4),(\frac{-1}{\cdot})) \longleftrightarrow$ (A-parameters)

$$F' \in S^+_{k-\frac{1}{2},j}(\Gamma_0(4),(\tfrac{-1}{\cdot})) \longleftrightarrow F \in J^{\star,\mathrm{cusp}}_{(k,j)}$$
$$\rightsquigarrow \pi_F \subset L^2_{\mathrm{disc}}(\mathrm{Sp}^J_4(\mathbb{Q})\backslash \mathrm{Sp}^J_4(\mathbb{A}_\mathbb{Q}))_\psi \longleftrightarrow \pi'_F \subset L^2_{\mathrm{disc}}(\mathrm{Mp}_4) \rightsquigarrow \phi_F$$

Lemma

We have $\phi_F = \tau_F \boxtimes S_1$,
$\exists \tau_F$: an irr. symplectic cuspidal autom. rep. of $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$ s.t.
$L(s,F') = L(s-j-k+\frac{3}{2},\tau_F)$ and $\tau_{F,\infty} = \mathcal{D}_{k+j-\frac{3}{2}} \oplus \mathcal{D}_{k-\frac{5}{2}}$.

Moreover, $\mathbb{C}F' \mapsto \tau_F$ is a bijection between
$\{$1-dim'l Hecke eigenspace in $S^+_{k-\frac{1}{2},j}(\Gamma_0(4),(\frac{-1}{\cdot}))\}$ and
$\{$irr. symp. cusp. autom. rep. of GL_4 that is unramified at any p and $\mathcal{D}_{k+j-\frac{3}{2}} \oplus \mathcal{D}_{k-\frac{5}{2}}$ at $\infty\}$.

$\because$) Rule out other possibilities by using the local information on $\pi'_{F,v}$ (and that j is even).
For the last assertion, do the converse. □

Sketch (again)

$$
\begin{array}{ccccc}
S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z})) & \rightsquigarrow & L^2_{\mathrm{disc}}(\mathrm{SO}_5(\mathbb{Q})\backslash \mathrm{SO}_5(\mathbb{A}_\mathbb{Q})) & \longleftrightarrow & (A\text{-parameters}) \\
\vert & & & & \Vert \\
S^+_{k-\frac{1}{2},j}(\Gamma_0(4), \left(\frac{-1}{\cdot}\right)) & & L^2_{\mathrm{disc}}(\mathrm{Mp}_4) & \longleftrightarrow & (A\text{-parameters}) \\
\downarrow \wr & & \updownarrow & & \\
J^{\star,\mathrm{cusp}}_{(k,j)} & \rightsquigarrow & L^2_{\mathrm{disc}}(\mathrm{Sp}^J_4(\mathbb{Q})\backslash \mathrm{Sp}^J_4(\mathbb{A}_\mathbb{Q}))_\psi & &
\end{array}
$$

Remark

$F \in S^+_{k-\frac{1}{2},j}(\Gamma_0(4), \left(\frac{-1}{\cdot}\right)) \rightsquigarrow \varphi_F \in \pi_F \subset L^2_{\mathrm{disc}}(\mathrm{Mp}_4)$

- level $\Gamma_0(4) \rightsquigarrow$ NOT $\mathrm{Sp}_4(\mathbb{Z}_2)$-invariant
- There does not exist the notion of spherical representations of $\mathrm{Mp}_4(\mathbb{Q}_2)$.

$\pi \subset L^2_{\mathrm{disc}}(\mathrm{Mp}_4) \rightsquigarrow F_\pi \in S_{k-\frac{1}{2},j}(\Gamma_0(4), \left(\frac{-1}{\cdot}\right))$

- difficult to see $F_\pi \in S^+_{k-\frac{1}{2},j}(\Gamma_0(4), \left(\frac{-1}{\cdot}\right))$

Theorem (Ibukiyama's conjectures)

Let $k \geq 3$, and $j \geq 0$ even. Then there exists an injective linear map

$$\mathcal{L} : S_{2k-4}(\mathrm{SL}_2(\mathbb{Z})) \otimes S_{2k+2j-2}(\mathrm{SL}_2(\mathbb{Z})) \hookrightarrow S^+_{k-\frac{1}{2},j}(\Gamma_0(4))$$
$$f \otimes g \mapsto \mathcal{L}(f \otimes g)$$

such that if f and g are Hecke eigenforms, then so is $\mathcal{L}(f \otimes g)$, and they satisfy

$$L(s, \mathcal{L}(f \otimes g)) = L(s-j-1, f)L(s, g).$$

Moreover, there exist linear isomorphisms

$$(\mathrm{Im}\,\mathcal{L})^\perp \cong S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z})) \cong S^+_{k-\frac{1}{2},j}(\Gamma_0(4), \left(\frac{-1}{\cdot}\right))$$

which preserve the L-functions.

$$S_{2k-4}(\mathrm{SL}_2(\mathbb{Z})) \otimes S_{2k+2j-2}(\mathrm{SL}_2(\mathbb{Z}))$$

$$S^+_{k-\frac{1}{2},j}(\Gamma_0(4)) = \qquad \mathrm{Im}\,\mathcal{L} \quad \oplus \quad (\mathrm{Im}\,\mathcal{L})^\perp$$

$$S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z}))$$

$$S^+_{k-\frac{1}{2},j}(\Gamma_0(4), \left(\frac{-1}{\cdot}\right))$$