GL(n)とその内部形式の正則超尖点表現の局所Langlands対応

(joint work with Masao Oi)

§ Introduction

F: p-adic field. G: conn. reductive gp. $/F$

Local Langlands correspondence for G (conjecture)

$\exists$ natural map w/ fin. fibers:

$$\mathrm{LLC}_G : \underbrace{\{\text{irr. smooth rep's of } G(F)\}/_{\cong}}_{\mathrm{Irr}(G)} \longrightarrow \underbrace{\{L\text{-parameters of } G\}/_{\sim}}_{\Phi(G)}$$

- Many partial constructions are known.

Today: Comparison of different approaches in 2 cases

Examples of LLC

① **GL_n case** (Harris-Taylor)

$$\mathrm{LLC}_{GL_n} : \mathrm{Irr}(GL_n) \xrightarrow[\sim]{1:1} \Phi(GL_n) \overset{\text{can.}}{\simeq} \left\{\begin{matrix}\text{n-dim. Frobenius semi-simple} \\ \text{Weil-Deligne rep's}\end{matrix}\right\}/_{\cong}$$

$$\begin{matrix} \cup & & \cup \\ \underbrace{\{\text{supercuspidal rep's}\}/_{\cong}}_{\mathrm{Irr}_{sc}(GL_n)} & \xrightarrow{\sim} & \{\text{n-dim. irr. sm. rep's of } W_F\}/_{\cong} \end{matrix}$$

W_F: Weil gp of F

- $\exists$ characterization using L-, ε-factors for pairs of rep's.

② **inner form case**

A: central simple alg. $/F$ s.t. $\dim_F A = n^2$

H: alg. gp. $/F$ s.t. $H(F) = A^\times$

($\rightsquigarrow$ H: inner form of GL_n, $\Phi(GL_n) = \Phi(H)$)

Local Jacquet-Langlands corr. (Deligne-Kazhdan-Vignéras)

$\exists$ nat. bij.

$\mathrm{LJLC}: \mathrm{Irr}_{ds}(GL_n) \xrightarrow{\sim} \mathrm{Irr}_{ds}(H)$ — ds: discrete series rep.

- $\mathrm{Irr}_{sc}(*) \subset \mathrm{Irr}_{ds}(*)$
- characterization: a character relation
- LLC_H is related by $LLC_{GL_n} = LLC_H \circ LJLC$

③ <u>regular sc. rep. case</u>

Assume G is tamely ram., $p \gg 0$

$(e.g.\ G = GL_n, H,\ p \neq 2)$

Kaletha defined $\mathrm{Irr}_{rsc}(G) := \{\text{reg. sc. rep's}\}/_{\cong} \subset \mathrm{Irr}_{sc}(G)$

and constructed

$$\left(\begin{array}{l}\text{if } G = GL_n, H, \\ GL_n(\mathbb{C}) \times W_F\end{array}\right)$$

$$LLC_G^{Kal} : \mathrm{Irr}_{rsc}(G) \longrightarrow \Phi(G),\ \pi \mapsto \phi : W_F \longrightarrow \underbrace{{}^L G}_{L\text{-gp.}} = \underbrace{\hat{G}}_{\text{dual gp.}/\mathbb{C}} \rtimes W_F$$

- $\mathrm{Irr}_{rsc}(G)$ is parametrized by certain pairs (S, ξ):

$$\begin{cases} S \subset G : \text{tame, elliptic maximal torus} \\ \xi : S(F) \to \mathbb{C}^\times \quad \text{"reg." char.} \end{cases}$$

(if $G = GL_n$ or H, then $S \cong \mathrm{Res}_{E/F}\mathbb{G}_m$ ($^\exists E/F$: tame, deg n))

$\rightsquigarrow \pi_{(S,\xi)}$: corresp. reg. sc. rep.

- LLC_G^{Kal} works uniformly for G.
- $\longmapsto$ is explicit and is mainly guided by a char. formula of π

Assume $p \neq 2$.

<u>Thm. 1</u> (Oi–T.) $^\forall \pi \in \mathrm{Irr}_{rsc}(GL_n)$

$$LLC_{GL_n}^{Kal}(\pi) = LLC_{GL_n}(\pi)$$

<u>Rem</u> If $p \nmid n$, then $\mathrm{Irr}_{rsc}(GL_n) = \mathrm{Irr}_{sc}(GL_n)$.

One can define

$$LJLC^{Kal} : \mathrm{Irr}_{rsc}(GL_n) \to \mathrm{Irr}_{rsc}(H)$$

by $LLC_{GL_n}^{Kal} = LLC_H^{Kal} \circ LJLC^{Kal}$

Thm. 2 (T.) $\forall \pi \in \mathrm{Irr}_{rsc}(GL_n)$

$$LJLC^{Kal}(\pi) = LJLC(\pi)$$

Rem Thm. 2 was already known if $p \geq (e(F/\mathbb{Q}_p)+2)\cdot n$

(Kaletha's arg. + Fintzen-Kaletha-Spice's char. formula)

Key Explicit description of $LLC_{GL_n}(\pi(S,\xi))$ & $LJLC(\pi(S,\xi))$

due to Bushnell-Henniart and Tam

(Still, comparing different conventions was not easy!)

From now on, fix $\pi = \pi(S,\xi)$ ($\rightsquigarrow S = \mathrm{Res}_{E/F}\mathbb{G}_m$, $\exists E/F$)

Focus on Thm. 1 for a while.

§ Bushnell-Henniart's theory

- For GL_n, $(S,\xi) \mapsto \pi(S,\xi)$ was classically known by Howe.
- ξ : char. of $S(F) = E^\times \overset{LCFT}{\longleftrightarrow}$ char. of W_E

$\rightsquigarrow$ naive guess: $LLC_{GL_n}(\pi(S,\xi)) \underset{??}{=} \underbrace{\mathrm{Ind}_{W_E}^{W_F}\xi}$ irr. by reg.

— No!

Thm (Bushnell-Henniart)

$\exists \mu_{rec}$: tamely ram. char. of $E^\times$ (dep. on ξ) "rectifier"

s.t. $LLC_{GL_n}(\pi(S,\xi)) = \mathrm{Ind}_{W_E}^{W_F}(\mu_{rec}^{-1}\cdot\xi)$

§ Kaletha's LLC

Q. How can one get an L-par. from (S,ξ)?

- By LLC for tori, $\xi : S(F) \to \mathbb{C}^\times$ induces $\phi_\xi : W_F \to {}^LS$

$\longrightarrow$ Want ${}^LS \to {}^LGL_n$

- $j : S \hookrightarrow GL_n$ induces $\hat{j} : \hat{S} \hookrightarrow \widehat{GL_n}$

$$\begin{tikzcd}
\hat{S} \arrow[r, hook, "\hat{j}"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & \widehat{GL_n} \arrow[d, hook] \\
\hat{S} \rtimes W_F = {}^L S \arrow[r, dotted, "\exists ?"] \arrow[dr] \arrow[r, phantom, shift right=4ex, "\circlearrowleft"] & {}^L GL_n = GL_n(\mathbb{C}) \times W_F \arrow[d] \\
& W_F
\end{tikzcd}$$

According to Langlands-Shelstad, a choice of χ-data χ (later) yields such a hom. ${}^L S \xrightarrow[{}^L j_\chi]{} {}^L GL_n$

In fact, Kaletha constructed

$$\left.\begin{cases} \chi_{Kal} : \chi\text{-data} \\ \varepsilon : S(F) \to \{\pm 1\}, \text{ char.} \end{cases}\right\} \text{ dep. on } (S, \xi)$$

and defined $LLC^{Kal}_{GL_n}(\pi_{(S,\xi)}) := {}^L j_{\chi_{Kal}} \circ \phi_{\varepsilon \cdot \xi}$

§ Tam's result

Prop. (Tam) S: as before.

χ: χ-data, $\rightsquigarrow$ ${}^\exists \mu_\chi$: char. of $E^\times$

s.t. ${}^L j_\chi \circ \phi_\xi \longleftrightarrow \mathrm{Ind}_{W_E}^{W_F}(\mu_\chi \cdot \xi)$ $({}^\forall \xi)$

$\rightsquigarrow LLC^{Kal}_{GL_n}(\pi_{(S,\xi)}) \longleftrightarrow \mathrm{Ind}_{W_E}^{W_F}(\mu_{\chi_{Kal}} \cdot \varepsilon \xi)$

Thm. (Tam)

${}^\exists \chi_{Tam}$: χ-data (dep. on ξ) s.t. $\mu_{\chi_{Tam}} = \mu_{rec}$

$\rightsquigarrow LLC_{GL_n}(\pi_{(S,\xi)}) = \mathrm{Ind}_{W_E}^{W_F}(\mu_{\chi_{Tam}}^{-1} \cdot \xi)$

Suffices to show $\mu_{\chi_{Kal}} \cdot \varepsilon = \mu_{\chi_{Tam}}^{-1}$ ——— ⊛

§ χ-data, μ_χ

$R := \{\text{roots of } S_{\overline{F}} \text{ in } GL_{n,\overline{F}}\} < \mathrm{Hom}(S_{\overline{F}}, \mathbb{G}_{m,\overline{F}}) \cong \mathbb{Z}^n$

$\Gamma_F, \{\pm 1\} \circlearrowright R$; $\quad \Gamma_F := \mathrm{Gal}(\overline{F}/F), \{\pm 1\} \circlearrowright \mathrm{Hom}(S_{\overline{F}}, \mathbb{G}_{m,\overline{F}})$

$\alpha \in R$, $\Gamma_\alpha := \mathrm{Stab}_{\Gamma_F}(\alpha) < \Gamma_{\pm\alpha} := \mathrm{Stab}_{\Gamma_F}(\{\pm\alpha\}) < \Gamma_F$

$\longleftrightarrow \quad F_\alpha > F_{\pm\alpha} > F$

Roughly,

- χ-data $\chi = (\chi_\alpha)_{\alpha \in R}$ is a set of char's

$\chi_\alpha : F_\alpha^\times \to \mathbb{C}^\times$ compatible w/ actions of $\Gamma_F, \{\pm 1\}$

- $\mu_\chi := \prod_{\substack{\Gamma_E \backslash R \\ \Downarrow \\ [\alpha]}} \chi_\alpha|_{E^\times}$, where $\alpha \in [\alpha]$ is chosen "suitably" (e.g. $E < F_\alpha$)

In fact, ε is also a product: $\varepsilon = \prod_{(\Gamma_E \times \{\pm 1\})\backslash R} \varepsilon_\alpha$ ($\exists\, \varepsilon_\alpha : E^\times \to \{\pm 1\}$ dep. only on the orb.)

$\rightsquigarrow$ For $\circledast$, suffices to show eq's in $\chi_{\mathrm{Kal},\alpha}$, ε_α, $\chi_{\mathrm{Tam},\alpha}$

§ Inner form case

Write ${}_{\mathrm{GL}_n}\chi_{\mathrm{Kal}}$, ${}_{H}\chi_{\mathrm{Kal}}$ ... to distinguish.

- ${}_{\mathrm{GL}_n}\chi_{\mathrm{Kal}} = {}_{H}\chi_{\mathrm{Kal}}$

$\rightsquigarrow$ $\mathrm{LJLC}^{\mathrm{Kal}}({}_{\mathrm{GL}_n}\pi(S, \xi)) = {}_{H}\pi(S, {}_{H}\varepsilon^{-1} \cdot {}_{\mathrm{GL}_n}\varepsilon \cdot \xi)$

- Thm (Bushnell-Henniart) $\exists\, \nu_{\mathrm{rec}}$: tame char. of $E^\times$

$\mathrm{LJLC}({}_{\mathrm{GL}_n}\pi(S, \xi)) = {}_{H}\pi(S, \nu_{\mathrm{rec}}^{-1}\xi)$

$\longrightarrow$ Suffices to show ${}_{H}\varepsilon^{-1} \cdot {}_{\mathrm{GL}_n}\varepsilon = \nu_{\mathrm{rec}}^{-1}$

- Thm (Tam) $\exists\, \zeta_{\mathrm{Tam}}$: "ζ-data"

$\nu_{\mathrm{rec}} = \prod_{[\alpha] \in \Gamma_E \backslash R} \zeta_{\mathrm{Tam},\alpha}|_{E^\times}$

$\longrightarrow$ Suffices to show eq's in ${}_{H}\varepsilon_\alpha$, ${}_{\mathrm{GL}_n}\varepsilon_\alpha$, $\zeta_{\mathrm{Tam},\alpha}$

§ Some more details

Def. of $\chi_{\mathrm{Kal},\alpha}$, ε_α, $\chi_{\mathrm{Tam},\alpha}$, $\zeta_{\mathrm{Tam},\alpha}$: case-by-case dep. on

α :
- asym. (i.e. $F_\alpha = F_{\pm\alpha}$) ①
- sym.
  - ur. (" $F_\alpha / F_{\pm\alpha}$ quad. ur.) ②
  - ram. (" " ram) ③

Rem
- ① and ② are simpler (char's are quad.)
- ③ is more subtle
  - char's can be of order 4
  - $\chi_{\mathrm{Kal},\alpha}$ involves $\lambda_{F_\alpha/F_{\pm\alpha}}$ (Langlands const.)

$\left(\chi_{\mathrm{Tam},\alpha} \text{ ---''--- } \text{Gauss sum of a quad. form.} \right.$

e.g. In ①, $\chi_{\mathrm{Kal},\alpha} = \mathbb{1}$

For ε_α, $\chi_{\mathrm{Tam},\alpha}$, $\xi_{\mathrm{Tam},\alpha}$, we need "$V_{[\alpha]}$"

$$\left[\begin{array}{l} (S,\xi) \rightsquigarrow V : \text{f.d. } k_F\text{-v.s. appearing in def. of } \pi_{(S,\xi)} \\ \qquad\qquad \| \\ \qquad \bigoplus_{[\alpha]\in\Gamma_F\backslash R} V_{[\alpha]} \\ V_{[\alpha]} \cong 0 \text{ or } k_{F_\alpha} \end{array}\right.$$

$$\varepsilon_\alpha := \begin{cases} \mathbb{1} & (\text{if } V_{[\alpha]} = 0) \\ \left(\frac{\alpha(\cdot)}{k_{F_\alpha}^\times}\right) : S(F) \xrightarrow{\alpha} \mathcal{O}_{F_\alpha}^\times \longrightarrow k_{F_\alpha}^\times \longrightarrow \{\pm 1\} & (\text{o/w}) \end{cases}$$

$$\begin{array}{ccc} S(F) & \xrightarrow{\alpha} & \mathcal{O}_{F_\alpha}^\times \\ \cap & & \cap \\ S(F_\alpha) & \xrightarrow{\alpha} & F_\alpha^\times \end{array}$$

$\chi_{\mathrm{Tam},\alpha}$, $\xi_{\mathrm{Tam},\alpha}$: related to the signature of a perm. of $V_{[\alpha]}$.