L 関数から生ずる正準系について

東京工業大学　鈴木正俊

早稲田整数論セミナー　2022年5月6日

・正準系, 擬正準系の逆問題

・逆問題の解法の具体例

・多項式

・ゼータ関数

・L関数

Quasi canonical systems

$t_0 < t_1 \leq \infty$

- $H(t) : I = [t_0, t_1) \to \mathrm{Sym}_2(\mathbb{R})$

$u(t, z) : \underset{t,\ z}{I \times \mathbb{C}} \longrightarrow \mathbb{C}^{2\times 1}$

Hamiltonian of QCS

$$0 = \frac{d}{dt} u(t, z) + z \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} H(t)\, u(t, z) \qquad z \in \mathbb{C}$$

is called a quasi canonical system on I.

QSC, QCS(H)

- QCS is called a canonical system (CS(H)) if

i) $H(t) \geq 0$ for a.e $t \in I$

ii) $H(t) \not\equiv 0$ on $\forall J \subset I$ w/. $|J| > 0$;

iii) $H(t) = (h_{ij}(t))$. $h_{ij}(t) \in L^1_{loc}(I)$

Some classical ODEs are reduced to CS.

eg. time-independent 1-dim Schrödinger eq

$$-y''(t) + q(t)\,y(t) = z\,y(t) \qquad q \in L^1_{loc}(I)$$

$\Rightarrow$ Let $-\alpha'' + q\alpha = 0$, $-\beta'' + q\beta = 0$ with $W = \alpha\beta' - \alpha'\beta \neq 0$

Define $A(t.z) = \frac{1}{W}\left(y(t.z)\beta'(t) - y'(t.z)\beta(t) \right)$

$B(t.z) = -\frac{1}{W}\left(y(t.z)\alpha'(t) - y'(t.z)\alpha(t) \right)$.

Then $u(t,z) := {}^t(A(t,z), B(t.z))$ solves CS(H) with

$$H(t) = \frac{1}{W}\begin{pmatrix} \alpha(t)^2 & \alpha(t)\beta(t) \\ \alpha(t)\beta(t) & \beta(t)^2 \end{pmatrix}.$$

Other examples : Dirac eq., string eq. etc.

More specifically, ...

$$\left(-\frac{\partial^2}{\partial t^2} + \left(\frac{1}{4} e^{2t} - k e^{t} + \frac{1}{4} \right) \right) \psi(t, z) = z^2 \psi(t, z)$$

$$\left(\left(\frac{\partial^2}{\partial x^2} + \left(-\frac{1}{4} + \frac{k}{x} + \frac{\frac{1}{4} - \mu^2}{x^2} \right) \right) w = 0, \quad x = e^t, \quad \mu = \pm\sqrt{\frac{1}{4} - z^2} \right)$$

$$\psi(t, z) = e^{-\frac{t}{2}} f(t, z)$$

$$f(t, z) = A\, W_{k,\mu}(e^t) + B\, M_{k,\mu}(e^t), \quad \mu = \pm\sqrt{\frac{1}{4} - z^2}$$

$\boxed{k = 0}$

$$\alpha(t) = e^{-\frac{t}{2}} W_{0,\frac{1}{2}}(e^t) = e^{-\frac{t}{2}} e^{-e^t/2}$$

$$\beta(t) = e^{-\frac{t}{2}} M_{0,\frac{1}{2}}(e^t) = e^{-\frac{t}{2}} \cdot 2\sinh(e^t)$$

$$W = \alpha\beta' - \alpha'\beta = 1$$

The most simple case

$I = [0, a)$, $0 < a < \infty$,

$$
\begin{cases}
\dfrac{d}{dt} u(t, z) + z \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} u(t, z) = 0 \\[2ex]
\lim\limits_{t \to a} u(t, z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{cases}
$$

$$\rightsquigarrow \quad u'' + z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u = 0$$

$$\rightsquigarrow$$

$$\Rightarrow \quad u(t, z) = \begin{pmatrix} \cos((a-t)z) \\ \sin((a-t)z) \end{pmatrix}$$

$$\cos((a-t)z) - i \sin((a-t)z) = e^{-i(a-t)z}$$

- If $u(t,z) = {}^t(A(t,z), B(t,z))$ solves a CS on $[t_0, t_1)$

with $\lim_{t \to t_1} u(t,z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

as fct of z

Then $E(t,\underline{\underline{z}}) = A(t,\underline{\underline{z}}) - i\,B(t,\underline{\underline{z}})$ belongs to $\mathbb{HB}$ $\forall t \in I$.

Hermite-Biehler

$\mathbb{HB}$ consists of entire functions satisfying

$$|E(\bar{z})| < |E(z)| \quad \forall z \in \mathbb{C}_+$$

$$\& \quad E(z) \neq 0 \quad \forall z \in \mathbb{R}$$

$\mathbb{C}_+ = \{\operatorname{Im} z > 0\}$

a generalization of exponentials

① $H \rightsquigarrow CS(H) \overset{\text{solve}}{\rightsquigarrow} E \in \mathbb{HB}$ (direct problem)

Q: Does any E come from H? (inverse problem)
$CS(H)$

A: Yes! L. de Branges (1960 ± ε)

$\downarrow$

- But in general it is difficult to determine H explicity from E.
- de Branges' inverse theorem is limited to $\mathbb{HB}$.

Q What's great about solving the inverse problem?
(for number theorists)

For $f(x) \in \mathbb{C}[x]$, we consider

$$E_f(z) := e^{irdz/2} f(e^{-irz})$$

$d = \deg f$, $r = \begin{cases} 1 & \deg f : \text{even} \\ 2 & \deg f : \text{odd} \end{cases}$ $\leftarrow$ for simplicity or by a technical reason.

◎ All roots of f lie inside $\mathbb{T} = \{|z|=1\}$

$\overset{\text{iff}}{\Longleftrightarrow}$ $E_f \in \mathbb{HB}$ $\rightsquigarrow$ $^\exists H = H_f$ of CS by de Branges

(elementaly)

$\uparrow$

what is the explicit form?

To state the 1st result, we set for

$$f(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0 \quad \in \mathbb{C}[X]$$

as

$$\underset{n\times n}{M_n(f)} = \begin{pmatrix} a_d & a_{d-1} & \cdots & \cdots & a_{d-n+1} \\ & a_d & a_{d-1} & \cdots & a_{d-n+2} \\ & & \ddots & \ddots & \vdots \\ & 0 & & \ddots & a_{d-1} \\ & & & & a_d \end{pmatrix}, \quad \underset{n\times n}{N_n(f)} = \begin{pmatrix} a_0 & a_1 & \cdots & \cdots & a_{n-1} \\ & a_0 & a_1 & \cdots & a_{n-2} \\ & & \ddots & \ddots & \vdots \\ & 0 & & \ddots & a_1 \\ & & & & a_0 \end{pmatrix}$$

$$\underset{2n\times 2n}{L_n^{\pm}(f)} = \begin{pmatrix} {}^t M_n(f) & \pm\, {}^t\overline{N_n(f)} \\ \pm N_n(f) & M_n(f) \end{pmatrix}$$

$$\mathbb{R} \ni D_n(f) := \det L_n^{\pm}(f) \quad \leftarrow \text{indep. of } \pm \qquad (1 \le n \le d)$$

$$D_0(f) := 1$$

Thm. 1 (ArXiv: 2106.04061)

$f(x) \in \mathbb{C}[x]$, $\deg f = d$. Suppose that $D_d(f) \neq 0$

∃ explicit formula in terms of $L_n^{\pm}(f)$

$$\Rightarrow \quad H_f(t) = \frac{1}{D_{n-1}(f)\, D_n(f)} \, \tilde{H}_{f,n}, \quad \frac{r(n-1)}{2} \le t < \frac{rn}{2}, \quad 1 \le n \le d$$

∃ ; Schur-Cohn ; > 0

total interval $[0, \frac{rd}{2})$

s.t. $E_f(z) = A(0,z) - i B(0,z)$ for the solution

$u(t,z) = {}^t(A(t,z), B(t,z))$ of $QCS(H_f)$

with $\lim_{t \to \frac{rd}{2}} u(t,z) = {}^t\left(\frac{1}{2}(E_f(0) + \overline{E_f(0)}), \frac{i}{2}(E_f(0) - \overline{E_f(0)})\right)$

⇒ by Schur-Cohn test (next page)

• $H_f(t) > 0$ on $[0, \frac{rd}{2})$ iff all roots of f are inside $\mathbb{H}$

i.e. This is H in the de Branges inverse thm.

Schur-Cohn test (Schur $\substack{1917\\1918}$, Cohn 1922)

Suppose that $D_n(f) \neq 0$ for all $1 \leq n \leq d$, and

$q :=$ # of sign changes in $(D_0(f), D_1(f), \cdots, D_d(f))$.

$\Rightarrow$ f has no roots on $\mathbb{T}$

has exactly $d-q$ roots inside $\mathbb{T}$ counting multiplicity.

In particular,

$$\text{all roots of } f \text{ are inside } \mathbb{T} \overset{\text{iff}}{\iff} D_n(f) > 0 \text{ for all } 1 \leq n \leq d.$$

χ : mod $q > 1$ primitive Dirichlet char.

$$\xi_\chi(s) = e^{-i\theta}\left(\frac{\pi}{q}\right)^{-\frac{s+\delta}{2}} \Gamma\left(\frac{s+\delta}{2}\right) L(s,\chi), \quad \delta = \begin{cases} 0 & \chi : \text{even} \\ 1 & \chi : \text{odd} \end{cases}$$

$\uparrow$

$\theta \in \mathbb{R}$ is chosen so that $\xi_\chi(s) = \overline{\xi_\chi(1-\bar{s})}$ holds.

Lagarias (2005)

All zeros of $\xi_\chi(s)$ lie on the line $\mathrm{Re}(s) = \frac{1}{2}$ & simple.

GRH

$\Longleftrightarrow$ $E_\chi(z) := \xi_\chi\left(\frac{1}{2} - iz\right) + \xi_\chi'\left(\frac{1}{2} - iz\right) \in \mathbb{HB}$

iff

$E_\chi \longleftrightarrow H_\chi$, $CS(H_\chi)$???

Instead of ξ_χ, we consider the family

$$E_\chi^{\omega,\nu}(z) = \xi_\chi\left(\tfrac{1}{2}+\omega-iz\right)^\nu, \quad (\omega,\nu)\in \mathbb{R}_{>0}\times\mathbb{Z}_{>0}$$

$$= \xi_\chi(s)^{\nu-1}\left[\xi_\chi\left(\tfrac{1}{2}-iz\right)+\omega\nu\,\xi_\chi'\left(\tfrac{1}{2}-iz\right)\right]+O_z(\omega^2)$$

$\rightsquigarrow$ GRH for $L(s,\chi) \overset{\text{iff}}{\Longleftrightarrow} E_\chi^{\omega,\nu}\in\mathbb{HB}$, ${}^\forall\omega>0$, ${}^\exists\nu=\nu(\omega)$

$E_\chi^{\omega,\nu}(z) \longleftrightarrow H_\chi^{\omega,\nu}(t)$?

$\uparrow$

Can we construct this without "$E_\chi^{\omega,\nu}\in\mathbb{HB}$".

$\Big\}$ Yes!

$$\Theta_x^{\omega,\nu}(z) := \overline{E_x^{\omega,\nu}(\bar{z})} \,/\, E_x^{\omega,\nu}(z)$$

$$K_x^{\omega,\nu}(x) := \frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic} \Theta_x^{\omega,\nu}(z)\, e^{-izx}\, dz, \quad c > \frac{1}{2}+\omega$$

$$\left[\, K_x^{\omega,\nu}(x) = 0 \text{ on } (-\infty, 0) \,\right]$$

For each $t \in \mathbb{R}$, we define

$$\mathcal{K}_x^{\omega,\nu}[t] : L^2(-\infty, t) \to L^2(-\infty, t) \quad \text{by}$$

$$(\mathcal{K}_x^{\omega,\nu}[t] f)(x) = \mathbb{1}_{(-\infty,t)}(x) \int_{-\infty}^{t} K_x^{\omega,\nu}(x+y)\, \overline{f(y)}\, dy.$$

- $\omega\nu > 1 \implies K_x^{\omega,\nu}$: continuous on $\mathbb{R}$.

  $\mathcal{K}_x^{\omega,\nu}[t]$: compact op.

Thm 2 (JFA 2020, 2021; TMJ 2022+ ; arXiv: 2012.11121)

. $\omega \in \mathbb{R}_{>0}$, $\nu \in \mathbb{Z}_{>0}$, $\omega\nu > 1$.

. Assume $\boxed{\text{GRH}}$ if $0 < \omega < \frac{1}{2}$. $\Longrightarrow$ For $\forall t \geq 0$, eq. on $L^2(-\infty, t)$

$$(1 \pm \mathbb{K}_x^{\omega,\nu}[t])\,\varphi^{\pm} = \mp \mathbb{K}_x^{\omega,\nu}[t]\,\mathbb{1}_{(-\infty,t)} \quad \text{has unique sol.}$$

Define $\Phi^{\pm}(t,x) := 1 \mp \int_{-\infty}^{t} K_x^{\omega,\nu}(x+y)\,\overline{(\varphi^{\pm}(t,y)+1)}\,dy$.

Then

$$\underbrace{H_x^{\omega,\nu}(t)}_{\det = 1} = \begin{pmatrix} |\Phi^+(t,t)|^2 & \Im(\Phi^+(t,t)\Phi^-(t,t)) \\ \Im(\Phi^+(t,t)\Phi^-(t,t)) & |\Phi^-(t,t)|^2 \end{pmatrix}$$

x: real

$$= \mathrm{diag}\left\{ \left(\frac{\det(1-\mathbb{K}_x^{\omega,\nu}[t])}{\det(1+\mathbb{K}_x^{\omega,\nu}[t])} \right)^2, \left(\frac{\det(1+\mathbb{K}_x^{\omega,\nu}[t])}{\det(1-\mathbb{K}_x^{\omega,\nu}[t])} \right)^2 \right\}$$

> 0 (because we assumed GRH)

Thm 3 (JFA 2020, 2021; TMJ 2022+; arXiv: 2012.11121)

GRH $\underset{\text{iff}}{\Longleftrightarrow}$ $\forall \omega > 0$, $\exists \nu = \nu(\omega) \in \mathbb{Z}_{>0}$ s.t

① $\omega\nu = 1$

② $1 \notin \sigma_p(\mathbb{K}_\chi^{\omega,\nu}[t])$

↖ the set of point spectrum (eigenvalues)

※ We can prove ② unconditionally if $t \geq 0$ is small enough.

($\xi_\chi(s) \neq 0$ for $\mathrm{Re}(s) > \frac{1}{2} + \omega_0$

$\underset{\text{iff}}{\Longleftrightarrow}$ $\forall \omega \geq \omega_0$, $\exists \nu = \nu(\omega) \in \mathbb{Z}_{>0}$ s.t ①, ② hold)

Thank you ! ▨