法p特異モジュラー形式の基底について
(On the basis of mod p singular forms)

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(On the basis of mod p singular forms)
Joint work with Prof. Boecherer

1. Introduction
2. Notation and Definition
3. Conjectures and Main Results
4. Proof of Theorem 1
5. Proof of Theorem 2
6. Proof of Theorem 3
7. Remarks and Problems

1. Introduction

1. Introduction

- Siegel modular group $\Gamma_n := \mathrm{Sp}_n(\mathbb{Z})$,
- $M_k(\Gamma_n)$: sp. of modular forms (MF) of wt k, for Γ_n,
- Fourier expansion:

$$F \in M_k(\Gamma_n) \quad \Longrightarrow \quad F = \sum_{0 \le T \in \Lambda_n} a_F(T)\boldsymbol{e}(\mathrm{tr}(TZ)),$$

$$\boldsymbol{e}(x) := e^{2\pi i x}, \qquad \Lambda_n := \{T = (t_{ij}) \in \mathrm{Sym}_n(\mathbb{Q}) \mid t_{ii} \in \mathbb{Z},\ 2t_{ij} \in \mathbb{Z}\}.$$

Definition

$$F \in M_k(\Gamma_n) : \textit{singular} \overset{\text{def}}{\Longleftrightarrow} a_F(T) = 0 \text{ for } \forall T \text{ with } \mathrm{rank}(T) = n.$$

$$r := \max\{\mathrm{rank}(T) \mid \exists T \text{ s.t. } a_f(T) \ne 0\}: \ \textit{(singular) rank}$$

1. Introduction

- Theta series $\quad 0 < S \in \Lambda_m,$

$$\theta_S^{(n)}(Z) := \sum_{X \in \mathbb{Z}^{m,n}} \boldsymbol{e}(\mathrm{tr}(S[X]Z)), \quad S[X] := {}^t XSX$$

: MF with some level of deg. n, wt $\dfrac{m}{2}$

$$A(S,T) := \sharp\{X \in \mathbb{Z}^{m,n} \mid S[X] = T\}$$

$$\Longrightarrow \quad \theta_S^{(n)} = \sum_{0 \le T \in \Lambda_n} A(S,T)\boldsymbol{e}(\mathrm{tr}(TZ))$$

In particular,

$$m < \mathrm{rank}(T) \quad \Longrightarrow \quad A(S,T) = 0 \quad \Longrightarrow \quad \theta_S^{(n)} : \text{ singular of rank } m$$

1. Introduction

Theorem (Freitag)

(1) $F \in M_k(\Gamma_n)$: singular $\iff$ $k < \dfrac{n}{2}$

(2) $F \in M_k(\Gamma_n)$: singular of rank r $\iff$ $k = \dfrac{r}{2}$

(3) $F \in M_k(\Gamma_n)$: singular of rank r $\Longrightarrow$ $f = \displaystyle\sum_{\substack{S\in\Lambda_r \\ \mathrm{level}(S)=1}} c_S \theta_S^{(n)}$

Level N version is

(3)' F: singular of rank r with level N $\Longrightarrow$ $F = \displaystyle\sum_{\substack{S\in\Lambda_r \\ \mathrm{level}(S)\mid N}} c_S \theta_S^{(n)}$

1. Introduction

How about the case of mod p?

About the mod p analog of

(1) $\cdots$ not done

F: mod p singular,
$\exists E \in M_{p-1}(\Gamma_n)$ s.t. $E \equiv 1 \bmod p$
$\Longrightarrow$ $FE \equiv F \bmod p$
$\Longrightarrow$ wt. increased

(2) $\cdots$ studied in 2016 (Boecherer-K)

F : mod p singular of p-rank r $\Longrightarrow$ $2k - r \equiv 0 \bmod p-1$

$\not\Longleftarrow$

$k = \dfrac{r}{2}$

Today's talk

About a mod p analog of (3)

2. Notation and Definition

2. Notation and Definition

- Siegel upper half space:

$$\mathbb{H}_n := \{ Z = X + iY \in M_n(\mathbb{C}) \mid Y > 0 \ (\text{pos. def.}) \}$$

- Siegel modular group Γ_n:

$$\Gamma_n := \mathrm{Sp}_n(\mathbb{Z}) = \{ M \in M_{2n}(\mathbb{Z}) \mid {}^t M J_n M = J_n \} \ \left(J_n := \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}\right)$$

$$= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A^t D - B^t C = 1_n,\ A^t B,\ C^t D \in \mathrm{Sym}_n(\mathbb{Z}) \right\}$$

- Congruence subgroup of level $N \in \mathbb{N}$:

$$\Gamma_0^{(n)}(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \mid C \equiv 0_n \bmod N \right\}$$

2. Notation and Definition

- Generalized fractional transformation:

$$M\langle Z\rangle := (AZ+B)(CZ+D)^{-1}, \qquad Z\in\mathbb{H}_n, \qquad M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in\Gamma_n$$

- $F:\mathbb{H}_n\longrightarrow\mathbb{C}$: holomorphic function, $M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in M_{2n}(\mathbb{Z})$,

$$\begin{aligned} F\mid_k M &= F\mid M \\ &:= (\det M)^{\frac{nk}{2}}\det(CZ+D)^{-k}F\left((AZ+B)(CZ+D)^{-1}\right)\end{aligned}$$

- F: **Siegel modular form** of wt. k, char. χ for $\Gamma_0^{(n)}(N)$

$$\overset{\text{def}}{\Longleftrightarrow} \qquad F|_kM=\chi(\det D)F \quad \text{for} \quad \forall M=\begin{pmatrix} * & * \\ * & D\end{pmatrix}\in\Gamma_0^{(n)}(N)$$

2. Notation and Definition

- Sp. of modular forms:

$$M_k(\Gamma_0^{(n)}(N),\chi) := \{ \text{ Siegel MF of wt. } k, \text{ char. } \chi \text{ for } \Gamma_0^{(n)}(N) \ \}$$

- For a ring $R \subset \mathbb{C}$,

$$M_k(\Gamma,\chi)_R := \{F \in M_k(\Gamma,\chi) \mid a_F(T) \in R \text{ for } \forall T \in \Lambda_n\},$$

- $0 < S \in \Lambda_m,\qquad L = \mathrm{level}(S) := \min\{N \in \mathbb{Z}_{\geq 1} \mid N(2S)^{-1} \in 2\Lambda_m\},$

$$\chi_S(d) = \mathrm{sign}(d)^{\frac{m}{2}} \left(\frac{(-1)^{\frac{m}{2}} \det 2S}{|d|} \right) \quad \Longrightarrow \quad \theta_S^{(n)} \in M_{m/2}(\Gamma_0(L),\chi_S)_{\mathbb{Z}}$$

- $\widetilde{M}_k(\Gamma,\chi) := \{\widetilde{F} \mid F \in M_k(\Gamma,\chi)_{\mathbb{Z}_{(p)}}\},\qquad \widetilde{F} = \sum_T \widetilde{a_F(T)}\boldsymbol{e}(\mathrm{tr}(TZ))$

2. Notation and Definition

Definition

$F \in M_k(\Gamma_0^{(n+r)}(N), \chi)_{\mathbb{Z}_{(p)}}$: *mod p singular* of p-rank r

$$\overset{\text{def}}{\Longleftrightarrow} \quad a_F(T) \equiv 0 \bmod p \text{ for } \forall T \in \Lambda_{n+r} \text{ with } \operatorname{rank}(T) > r,$$

$$\exists T \in \Lambda_{n+r} \text{ with } \operatorname{rank}(T) = r \text{ s.t. } a_F(T) \not\equiv 0 \bmod p$$

By our result (2016), if $\chi^2 = 1$

$$F : \text{mod } p \text{ singular of } p\text{-rank } r \implies 2k - r \equiv 0 \bmod p-1$$

➡ r: even!

3. Conjectures and Main Results

3. Conjectures and Main Results

Conjecture 1

Any mod p singular form would be congruent mod p to a true singular form.

Conjecture 2

$n, r, k, N \in \mathbb{N}$, p: prime, χ: Dirichlet char. mod N s.t. $\chi^2 = 1$.

$$F \in M_k(\Gamma_0^{(n+r)}(N), \chi)_{\mathbb{Z}_{(p)}}: \text{ mod } p \text{ singular of } p\text{-rk } r \quad \Longrightarrow \quad \exists e \in \mathbb{N} \text{ s.t.}$$

$$F \equiv \sum_{\substack{T \in \Lambda_r / \mathrm{GL}_r(\mathbb{Z}) \\ \mathrm{level}(T) | p^e N}} c_T \vartheta_T^{(n+r)} \bmod p, \quad c_T \in \mathbb{Z}_{(p)}.$$

Here e is determined by k, like as, if k is small, then e is also small.

3. Conjectures and Main Results

The possible smallest weight case:

Conjecture 3

$n, r, k, N \in \mathbb{N}$, p: prime, χ: Dirichlet char. mod N s.t. $\chi^2 = 1$,

$$k = k(r,p) := \begin{cases} \frac{p+r-1}{2} & \text{if} \quad r \equiv 2 \bmod 4, \quad p \equiv -1 \bmod 4 \\ p - 1 + \frac{r}{2} & \text{if} \quad r \equiv 0 \bmod 4, \end{cases}$$

$F \in M_k(\Gamma_0^{(n+r)}(N), \chi)_{\mathbb{Z}_{(p)}}$: mod p singular of p-rk r

$$\Longrightarrow \quad F \equiv \sum_{\substack{T \in \Lambda_r / \mathrm{GL}_r(\mathbb{Z}) \\ \mathrm{level}(T) | pN}} c_T \vartheta_T^{(n+r)} \bmod p, \quad c_T \in \mathbb{Z}_{(p)}.$$

3. Conjectures and Main Results

Theorem 1 (Boecherer-K)

$n \geq r, \quad p-1 \nmid r,$

χ: Dirichlet char. mod N s.t. $\chi^2 = 1, \quad \chi(-1) = (-1)^k$

$F \in M_k(\Gamma_0^{(n+r)}(N), \chi)$: mod p singular of p-rk r,

$\Longrightarrow$ Conjecture 1 is ture, i.e.,

$\exists \{S_1, \cdots, S_t\} \subset \Lambda_r^+$: fin. set s.t. $\quad F \equiv \sum_{j=1}^{t} c_j \theta_{S_j}^{(n+r)} \bmod p$

Remark
- Actually we have $\theta_{S_j}^{(n)} \in \widetilde{M}_k(\Gamma_0^{(n)}(N), \chi)$.
- We do not mention anything about the level of S_j!

Theorem 2 (Boecherer-K)

$n \geq r, \quad p-1 \nmid r, \quad N=1,$

$F \in M_k(\Gamma_{n+r})$: mod p singular of p-rk r

$\Longrightarrow$ Conjecture 2 is ture, i.e.,

$\exists\ \{S_1, \cdots, S_t\} \subset \Lambda_r^+$: fin. set s.t. level$(S_j)$ is p-power and

$$F \equiv \sum_{j=1}^{t} c_j \theta_{S_j}^{(n+r)} \bmod p$$

3. Conjectures and Main Results

Theorem 3 (Boecherer-K)

$$n \geq 1, \quad r = 2, \quad p > n+5, \quad p \equiv -1 \bmod 4, \quad N = 1$$

$F \in M_{\frac{p+1}{2}}(\Gamma_{n+2})$: mod p singular of p-rk 2

$\Longrightarrow$ Conjecture 3 is ture, i.e.,

$\exists \{S_1, \cdots, S_t\} \subset \Lambda_2^+$: fin. set s.t. $\underline{\mathrm{level}(S_j) = p}$ and

$$F \equiv \sum_{j=1}^{t} c_j \theta_{S_j}^{(n+r)} \bmod p$$

4. Proof of Theorem 1

4. Proof of Theorem 1

Tool 1: Sturm set for mod p singular

$\Gamma \subset \Gamma_{n+r}$: a modular group of degree $n+r$

$\widetilde{M}_{k,r}^{p\text{-sing}}(\Gamma,\chi)$: sp. / $\mathbb{F}_p$ of mod p singular of p-rank $\leq r$

➡ Consider the quotient space

$$V = V_{k,r} := \widetilde{M}_k(\Gamma,\chi)/\widetilde{M}_{k,r}^{p\text{-sing}}(\Gamma,\chi)$$

$$\dim_{\mathbb{F}_p} M_k(\Gamma,\chi) < \infty \quad \Longrightarrow \quad \dim_{\mathbb{F}_p} V < \infty$$

➡ By the Linear Algebra, $\dim_{\mathbb{F}_p} V^* = \dim_{\mathbb{F}_p} V < \infty$

Here $V^* := \{\phi \mid \phi : V \to \mathbb{F}_p : \text{linear map}\}$

4. Proof of Theorem 1

➡ For $T \in \Lambda_{n+r}$ with $\mathrm{rank}(T) > r$, define $\ell_T : V \to \mathbb{F}_p$ by

$$\ell_T(F + \widetilde{M}_{k,r}^{p\text{-sing}}(\Gamma)) := \widetilde{a_F(T)}$$

$$\Longrightarrow \quad L := \langle \ell_T \mid T \in \Lambda_{n+r},\ \mathrm{rank}(T) > r \rangle_{\mathbb{F}_p} \subset V^*$$

$$\dim V^* < \infty \quad \Longrightarrow \quad \dim_{\mathbb{F}_p} L =: d < \infty$$

➡ We can select a basis

$$\langle \ell_{T_1}, \cdots, \ell_{T_d} \rangle_{\mathbb{F}_p} = L$$

➡ Hence we obtain a "Sturm set"

$$\mathcal{T}_{n+r,r} = \mathcal{T}_{n+r,r}(k,\Gamma) := \{T_1, \cdots, T_d\} :$$

4. Proof of Theorem 1

$$F \in M_k(\Gamma, \chi), \quad a_F(T) \equiv 0 \bmod p \quad \text{for} \quad \forall T \in \mathcal{T}_{n+r,r}$$

$$\Longrightarrow \quad F \in M_{k,r}^{p\text{-sing}}(\Gamma, \chi)$$

$\because$ $T \in \Lambda_{n+r}$ with $\mathrm{rk}(T) > r \quad \Longrightarrow$

$$a_F(T) = \ell_T(F) = \sum_{i=1}^{d} c_i \ell_{T_i}(F) = \sum_{i=1}^{d} a_F(T_i) = 0 \quad \text{in} \quad \mathbb{F}_p.$$

Notation

- $M_n^{(r)}(\mathbb{Z}) := \{ M \in M_n(\mathbb{Z}) \mid \mathrm{rank}(M) = r \}, \quad M_n^*(\mathbb{Z}) := M_n^{(n)}(\mathbb{Z})$
- $a(S) := a_F\left(\begin{smallmatrix} 0 & 0 \\ 0 & S \end{smallmatrix}\right)$ when $S \in \Lambda_r$

4. Proof of Theorem 1

- $a^*(T)$: the primitive Fourier coefficient defined by

$$a(T) = \sum_{\substack{G \in \mathrm{GL}_r(\mathbb{Z})\backslash M_r^*(\mathbb{Z}) \\ T[G^{-1}] \in \Lambda_r}} a^*(T[G^{-1}])$$

$$\Longrightarrow \quad a(T) = \sum_{S \in \Lambda_r^+/\mathrm{GL}_r(\mathbb{Z})} \frac{1}{\epsilon(S)} \sum_{\substack{V \in M_r^*(\mathbb{Z}) \\ S[V]=T}} a^*(S), \qquad \epsilon(S) := A(S,S)$$

Introduced by Boecherer-Raghavan.
We use this version!

- $F_{[r]} := \sum_{\substack{T \in \Lambda_n \\ \mathrm{rank}(T)=r}} a(T)\boldsymbol{e}(\mathrm{tr}(TZ))$

- $\Lambda_n^{(r)} := \{T \in \Lambda_n \mid \mathrm{rank}(T) = r\}, \quad \Lambda_n^+ := \Lambda_n^{(n)}$

4. Proof of Theorem 1

Variant of Freitag's argument

Lemma 4.1

$$F \in M_k(\Gamma_0^{(n+r)}(N), \chi), \quad \chi(-1) = (-1)^k$$

$$\Longrightarrow \quad F_{[r]}(Z) = \sum_{S \in \Lambda_r^+ / \mathrm{GL}_r(\mathbb{Z})} \frac{a^*(S)}{\epsilon(S)} \sum_{\substack{(X_1, X_2) \in \mathbb{Z}^{r,r} \times \mathbb{Z}^{r,n} \\ \mathrm{rank}(X_1, X_2) = r}} \boldsymbol{e}(\mathrm{tr}(S[(X_1, X_2)]Z)).$$

Note that

$$\sum_{(X_1, X_2) \in \mathbb{Z}^{r,r} \times \mathbb{Z}^{r,n}} \boldsymbol{e}(\mathrm{tr}(S[(X_1, X_2)]Z)) = \sum_{X \in \mathbb{Z}^{r,n+r}} \boldsymbol{e}(\mathrm{tr}(S[X]Z)) = \theta_S^{(r)}(Z).$$

➡ $$F_{[r]}(Z) = \sum_{S \in \Lambda_r^+ / \mathrm{GL}_r(\mathbb{Z})} \frac{a^*(S)}{\epsilon(S)} (\theta_S^{(n+r)}(Z))_{[r]}.$$

Proof.

For $X_1 \in \mathbb{Z}^{r,r}$, $X_2 \in \mathbb{Z}^{r,n}$, $\exists W \in M_r^*(\mathbb{Z})$ s.t.

$\exists ! W \in M_r^*(\mathbb{Z})/\mathrm{GL}_r(\mathbb{Z})$

$$(X_1, X_2) = W(G_1, G_2), \quad \begin{pmatrix} * & * \\ G_1 & G_2 \end{pmatrix} \in \mathrm{GL}_{n+r}(\mathbb{Z}).$$

RHS

$= a(T)$ if we put $S[W] = T$

$= T$

$$= \sum_S \frac{a^*(S)}{\epsilon(S)} \sum_{W \in M_r^*(\mathbb{Z})} \sum_{\substack{(G_1,G_2) \in \mathrm{GL}_r(\mathbb{Z}) \backslash \mathbb{Z}^{r,r} \times \mathbb{Z}^{r,n} \\ \begin{pmatrix} * & * \\ G_1 & G_2 \end{pmatrix} \in \mathrm{GL}_{n+r}(\mathbb{Z})}} \boldsymbol{e}(\mathrm{tr}(S[W][(G_1, G_2)]Z)),$$

$$\left(\begin{array}{l} \mathrm{GL}_r(\mathbb{Z}) \backslash \mathbb{Z}^{r,r} \times \mathbb{Z}^{r,n} = \sim \backslash \mathbb{Z}^{r,r} \times \mathbb{Z}^{r,n}, \\ (X_1, X_2) \sim (Y_1, Y_2) \overset{\mathrm{def}}{\Longleftrightarrow} \exists G \in \mathrm{GL}_r(\mathbb{Z}) \text{ s.t. } (X_1, X_2) = G(Y_1, Y_2) \end{array} \right.$$

4. Proof of Theorem 1

If we put $S[W]=T$, then

$$\mathrm{RHS}=\sum_{T\in\Lambda_r^+}a(T)\sum_{\substack{(G_1,G_2)\in\mathrm{GL}_r(\mathbb{Z})\backslash\mathbb{Z}^{r,r}\times\mathbb{Z}^{r,n}\\ \left(\begin{smallmatrix}* & *\\ G_1 & G_2\end{smallmatrix}\right)\in\mathrm{GL}_{n+r}(\mathbb{Z})}}\boldsymbol{e}(\mathrm{tr}(T[(G_1,G_2)]Z))$$

$$=\sum_{T\in\Lambda_r^+}\sum_{\substack{(G_1,G_2)\in\mathrm{GL}_r(\mathbb{Z})\backslash\mathbb{Z}^{r,r}\times\mathbb{Z}^{r,n}\\ \left(\begin{smallmatrix}* & *\\ G_1 & G_2\end{smallmatrix}\right)\in\mathrm{GL}_{n+r}(\mathbb{Z})}}a(T)\boldsymbol{e}(\mathrm{tr}(T[(G_1,G_2)]Z))$$

$$U:=\left(\begin{smallmatrix}* & *\\ G_1 & G_2\end{smallmatrix}\right)\Longrightarrow T[(G_1,G_2)]=\left(\begin{smallmatrix}0 & 0\\ 0 & T\end{smallmatrix}\right)[U]\Longrightarrow a(T)=a\left(\begin{smallmatrix}0 & 0\\ 0 & T\end{smallmatrix}\right)=a(T[(G_1,G_2)])$$

$$=\sum_{T}\sum_{\substack{(G_1,G_2)\in\mathrm{GL}_r(\mathbb{Z})\backslash\mathbb{Z}^{r,r}\times\mathbb{Z}^{r,n}\\ \left(\begin{smallmatrix}* & *\\ G_1 & G_2\end{smallmatrix}\right)\in\mathrm{GL}_{n+r}(\mathbb{Z})}}a(T[(G_1,G_2)])\boldsymbol{e}(\mathrm{tr}(T[(G_1,G_2)]Z))$$

4. Proof of Theorem 1

If T and (G_1, G_2) run, then $T[(G_1, G_2)]$ runs over $\Lambda_{n+r}^{(r)}$.

$$\text{RHS} = \sum_{T \in \Lambda_{n+r}^{(r)}} a(T)\boldsymbol{e}(\mathrm{tr}(TZ)) = F_{[r]}(Z) \quad \square$$

Let $Z_1 \in \mathbb{H}_r$, $Z_2 \in \mathbb{H}_n$.

$$F\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} = \sum_{\substack{\mathfrak{T} = \begin{pmatrix} T & * \\ * & * \end{pmatrix} \in \Lambda_{n+r} \\ T \in \Lambda_r}} \phi_T(Z_2)\mathbf{e}(\mathrm{tr}(TZ_1))$$

$$\Longrightarrow \quad \phi_T \in M_k(\Gamma_0^{(n)}(N), \chi) \quad \text{for} \quad \forall T \in \Lambda_r.$$

We denote by F^* the subseries of F, characterized by

$$\mathfrak{T} = \begin{pmatrix} T & * \\ * & * \end{pmatrix} \quad \text{with} \quad T \in \Lambda_r^+.$$

4. Proof of Theorem 1

$$\Rightarrow \quad F^*\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} = \sum_{\substack{\mathfrak{T}=\begin{pmatrix} T & * \\ * & * \end{pmatrix}\in\Lambda_{n+r} \\ T\in\Lambda_r^+}} \phi_T(Z_2)\mathbf{e}(\mathrm{tr}(TZ_1))$$

$$(\text{Still} \quad \phi_T \in M_k(\Gamma_0^{(n)}(N),\chi))$$

Suppose now that F is mod p singular of p-rank r.

$$\Rightarrow \quad F^*\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \equiv F^*_{[r]}\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \bmod p$$

$$\mathrm{RHS} = \sum_S \frac{a^*(S)}{\epsilon(S)} \sum_{\substack{(X_1,X_2)\in\mathbb{Z}^{r,r}\times\mathbb{Z}^{r,n} \\ \mathrm{rank}(X_1)=r}} \boldsymbol{e}(\mathrm{tr}(S[X_1]Z_1))\boldsymbol{e}(\mathrm{tr}(S[X_2]Z_2))$$

By Lemma 4.1

$*$: $\mathrm{rank}(X_1, X_2) = r \to \mathrm{rank}(X_1) = r$

$$= \sum_S \frac{a^*(S)}{\epsilon(S)} \sum_{\substack{X_1\in\mathbb{Z}^{r,r} \\ \mathrm{rank}(X_1)=r}} \boldsymbol{e}(\mathrm{tr}(S[X_1]Z_1)) \sum_{X_2\in\mathbb{Z}^{r,n}} \boldsymbol{e}(\mathrm{tr}(S[X_2]Z_2))$$

4. Proof of Theorem 1

$$= \sum_{S} \frac{a^*(S)}{\epsilon(S)} (\theta_S^{(r)}(Z_1))_{[r]} \theta_S^{(n)}(Z_2)$$

$$= \sum_{S \in \Lambda_r^+/GL_r(\mathbb{Z})} \frac{a^*(S)}{\epsilon(S)} \left(\sum_{T \in \Lambda_r^+} A(S,T) \boldsymbol{e}(\mathrm{tr}(TZ_1)) \right) \theta_S^{(n)}(Z_2)$$

$$= \sum_{T \in \Lambda_r^+} \left(\sum_{S \in \Lambda_r^+/GL_r(\mathbb{Z})} \frac{a^*(S)}{\epsilon(S)} A(S,T) \theta_S^{(n)}(Z_2) \right) \boldsymbol{e}(\mathrm{tr}(TZ_1))$$

$$\Rightarrow \quad \phi_T(Z_2) \equiv \sum_{S \in \Lambda_r^+/GL_r(\mathbb{Z})} A(T,S) \frac{a^*(S)}{\epsilon(S)} \theta_S^{(n)}(Z_2) \bmod p$$

We could see that

F is mod p singular of p-rank r $\Longrightarrow$ $\displaystyle\sum_{S \in \Lambda_r^+/GL_r(\mathbb{Z})} A(T,S) \frac{a^*(S)}{\epsilon(S)} \theta_S^{(n)}(Z_2) \in \widetilde{M}_k(\Gamma_0^{(n)}(N), \chi)$

(By Minkowski, $p-1 \nmid r \Longrightarrow p \nmid \varepsilon(S) = A(S,S)$)

4. Proof of Theorem 1

Proposition 4.2

p: prime with $p-1 \nmid r$,

$F \in M_k(\Gamma_0^{(n+r)}(N), \chi)_{\mathbb{Z}_{(p)}}$: mod p singular of p-rank r, $\quad \chi(-1) = (-1)^k$,

$S \in \Lambda_r^+$ with $a^*(S) \not\equiv 0$ mod $p \quad \Longrightarrow \quad \theta_S^{(n)} \in \widetilde{M}_k(\Gamma_0^{(n)}(N), \chi)$

Proof. Suppose that $\exists S$ s.t. the claim is not true.

Take S_0 in such S s.t. $\det S_0$ is minimal.

Consider

$$\phi_{S_0} - \sum_{\substack{S \in \Lambda_r^+/\mathrm{GL_r}(\mathbb{Z}) \\ \det S < \det S_0}} A(S, S_0) \frac{a^*(S)}{\epsilon(S)} \theta_S^{(n)} \in \widetilde{M}_k(\Gamma_0^{(n)}(N), \chi)$$

Then this becomes

$$\sum_{\substack{S\in\Lambda_r^+/\mathrm{GL}_r(\mathbb{Z})\\ \det S\geq\det S_0}} A(S,S_0)\frac{a^*(S)}{\epsilon(S)}\theta_S^{(n)} \equiv A(S_0,S_0)\frac{a^*(S_0)}{\epsilon(S_0)}\theta_{S_0}^{(n)}.$$

$S \not\sim S_0 \Longrightarrow A(S,S_0)=0$

$$= a^*(S_0)\theta_{S_0}^{(n)}$$

$\theta_{S_0}^{(n)} \in \widetilde{M}_k(\Gamma_0^{(n)}(N),\chi)$ Contradiction. $\square$

Corollary 4.3

$p-1\nmid r,\quad S\in\Lambda_r^+$: $a_F(S)\not\equiv 0 \bmod p$

$\det S$ is minimal. $\Longrightarrow$ $\theta_S^{(n)} \in \widetilde{M}_k(\Gamma_0^{(n)}(N),\chi)$

Remark. This property can be proved by also Freitag's original argument.

4. Proof of Theorem 1

We prove Theorem 1.

Theorem 1 (Boecherer-K)

$n \geq r, \quad p-1 \nmid r,$

χ: Dirichlet char. mod N s.t. $\chi^2 = 1, \quad \chi(-1) = (-1)^k$

$F \in M_k(\Gamma_0^{(n+r)}(N), \chi)$: mod p singular of p-rk r,

$\Longrightarrow$ Conjecture 1 is ture, i.e.,

$\exists \{S_1, \cdots, S_t\} \subset \Lambda_r^+$: fin. set s.t. $\quad F \equiv \sum_{j=1}^{t} c_j \theta_{S_j}^{(n+r)} \bmod p$

4. Proof of Theorem 1

Proof.

For the Sturm set $\mathcal{T}_{r,r-1} \subset \Lambda_r^+$ corresponding to $M_{k,r-1}^{p\text{-sing}}(\Gamma_0^{(r)}(N), \chi)$,

we take $M \in \mathbb{N}$ s.t.

$$M > \max\{\det T \mid T \in \mathcal{T}_{r,r-1}\}.$$

➡ Consider $G := F - \sum_{\substack{S \in \Lambda_r^+ \\ \det S < M}} \frac{a^*(S)}{\epsilon(S)} \theta_S^{(n+r)}$ and $g := \Phi^n(G)$.

(Φ: Siegel Φ-operator)

➡ $$g = \Phi^n(F) - \sum_{\substack{S \in \Lambda_r^+ \\ \det S < M}} \frac{a^*(S)}{\epsilon(S)} \theta_S^{(r)} \in \widetilde{M}_k(\Gamma_0^{(r)}(N), \chi)$$

By Prop 4.2 and $n \geq r$

We prove $g \in M_{k,r-1}^{p\text{-sing}}(\Gamma_0^{(r)}(N), \chi)$.

4. Proof of Theorem 1

It suffices to check the Fourier coefficients $a_g(T)$ for all $T \in \mathcal{T}_{r,r-1} \subset \Lambda_r^+$.

➡ $$a_g(T) = \sum_{\substack{S \in \Lambda_r^+ \\ \det S \geq M}} \frac{a^*(S)}{\epsilon(S)} A(S,T) \quad \text{for} \quad \forall T \in \Lambda_r^+$$

$$g = \Phi^n(F) - \sum_{\substack{S \in \Lambda_r^+ \\ \det S < M}} \frac{a^*(S)}{\epsilon(S)} \theta_S^{(r)}, \qquad F_{[r]} = \sum_{S \in \Lambda_r^+} \frac{a^*(S)}{\epsilon(S)} (\theta_S^{(n+r)}(Z))_{[r]}$$

$$S[X] = T \text{ with } X \in \mathbb{Z}^{r,r} \quad \Longrightarrow \quad \det S[X] = \det S (\det X)^2 = \det T$$

$$\Longrightarrow \quad \det S \leq \det T$$

➡ $$A(S,T) = 0 \quad \text{for} \quad \forall T \quad \text{with} \quad \det T < \det S \quad (\forall T \text{ with } \det T < M)$$

➡ $$g \in M_{k,r-1}^{p\text{-sing}}(\Gamma_0^{(r)}(N), \chi)$$

4. Proof of Theorem 1

Suppose that $g \not\equiv 0 \bmod p$.

$$\begin{cases} r:\ p\text{-rank of } F & \Longrightarrow \quad 2k - r \equiv 0 \bmod p-1 \\ r'\ (\leq r-1):\ p\text{-rank of } g & \Longrightarrow \quad 2k - r' \equiv 0 \bmod p-1 \end{cases}$$

By result of Boecherer-K

➡ $r - r' \equiv 0 \bmod p-1$

Since $p-1 > r > r'$, this is impossible!

➡ $g \equiv 0 \bmod p$ ➡ $\Phi^n(F) \equiv \sum_{\substack{S \in \Lambda_r^+ \\ \det S < M}} \frac{a^*(S)}{\epsilon(S)} \theta_S^{(r)} \bmod p$

$\because$ F is of p-rk r, $\theta_S^{(n+r)}$: singular of rk r

➡ $F \equiv \sum_{\substack{S \in \Lambda_r^+ \\ \det S < M}} \frac{a^*(S)}{\epsilon(S)} \theta_S^{(n+r)} \bmod p$

5. Proof of Theorem 2

5. Proof of Theorem 2

Tool 2: Strong q-expansion principle

Theorem (Katz (n=1), Ichikawa (n general))

$$F \in M_k(\Gamma_1^{(n)}(N))_{\mathbb{Z}_{(p)}} \ (p \nmid N) \quad \Longrightarrow \quad F|\gamma \text{ is } p\text{-integral for } \forall\, \gamma \in \Gamma_n$$

⬇

Theorem A (Boecherer-K)

$$F \in M_k(\Gamma_1^{(n)}(N) \cap \Gamma_0^{(n)}(p^m))_{\mathbb{Z}_{(p)}} \ (p \nmid N)$$

$$\Longrightarrow \quad F|\gamma \text{ is } p\text{-integral for } \forall\, \gamma \in \Gamma_0^{(n)}(p^m)$$

Tool 3: Formula for theta series

Theorem (Kitaoka (n=1))

$$0 < S \in \Lambda_n, \quad S \leftrightarrow L \text{ (lattice)}, \quad \mathrm{level}(S) = N,$$

$$d \mid N, \quad (d, \tfrac{N}{d}) = 1, \quad c := \tfrac{N}{d}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$$

$$\Longrightarrow \quad \theta_L^{(1)} \mid M = \kappa \cdot \theta_{L'}^{(1)}, \; L' \otimes \mathbb{Z}_p = \begin{cases} L \otimes \mathbb{Z}_p & \text{if} \quad p \nmid d \\ (L \otimes \mathbb{Z}_p)^* & \text{if} \quad p \mid d \end{cases}$$

5. Proof of Theorem 2

Theorem B (Boecherer-K)

$$0 < S \in \Lambda_n, \quad S \leftrightarrow L \text{ (lattice)}, \quad \mathrm{level}(S) = N,$$

$$d \mid N, \quad (d, \tfrac{N}{d}) = 1, \quad c := \tfrac{N}{d}, \quad \mathfrak{M} := \begin{pmatrix} a \cdot 1_n & b \cdot 1_n \\ c \cdot 1_n & d \cdot 1_n \end{pmatrix} \in \Gamma_n,$$

$$\Longrightarrow \quad \theta_L^{(n)} \mid \mathfrak{M} = \kappa \cdot \theta_{L'}^{(n)}, \qquad L' \otimes \mathbb{Z}_p = \begin{cases} L \otimes \mathbb{Z}_p & \text{if} \quad p \nmid d \\ (L \otimes \mathbb{Z}_p)^* & \text{if} \quad p \mid d \end{cases}$$

5. Proof of Theorem 2

As an application of these two theorems, we have

Proposition 5.1

p: prime with $p-1 \nmid n$,

$S \in \Lambda_n^+$, $\quad \mathrm{level}(S) = N = N' \cdot q^t$, $\quad q \neq p$, $\quad q^t || N'$ $(t > 0)$

$$\Longrightarrow \quad \theta_S^{(n)} \not\equiv \phi \bmod p \quad \text{for} \quad \forall \phi \in M_k(\Gamma_n)_{\mathbb{Z}_{(p)}}, \forall k$$

Proof. Suppose that $\exists \phi$ s.t. $\phi \equiv \theta_S^{(n)} \bmod p$.

Apply Theorem B to $\theta_S^{(n)}$ with $M = \begin{pmatrix} * & * \\ N' & q^t \end{pmatrix}$.

➡ $\theta_S^{(n)} \mid \mathfrak{M} = \kappa \cdot \theta_{S'}^{(n)} \equiv \phi \bmod p$. Here $L \leftrightarrow S$, $L' \leftrightarrow S'$.

By Theorem A

5. Proof of Theorem 2

➡ S': rational symmetric matrix with q appearing in its denominator

➡ $\kappa \cdot \theta_{S'}^{(n)}$ has S'th Fourier coefficient $\kappa \cdot A(S', S') \not\equiv 0 \bmod p$

From $p \nmid \kappa$ and
$p-1 \nmid n \Longrightarrow p \nmid A(S', S')$

➡ $a_\phi(S') \not\equiv 0 \bmod p$

➡ Since S' is not half integral, this is a contradiction! □

Corollary 5.2

p: prime with $p-1 \nmid n$, $S \in \Lambda_n^+$,

$\exists \phi \in M_k(\Gamma_n)_{\mathbb{Z}_{(p)}}$ s.t. $\theta_S^{(n)} \equiv \phi \bmod p \quad \Longrightarrow \quad \text{level}(S)$ is p-power

5. Proof of Theorem 2

Hence, we can restate Theorem 1 for $N=1$ as Theorem 2.

(Use $\theta_{S_j}^{(n)} \in \widetilde{M}_k(\Gamma_n) \implies \text{level}(S_j)$ is p-power)

Theorem 2 (Boecherer-K)

$n \geq r, \quad p-1 \nmid r, \quad N=1$

$F \in M_k(\Gamma_{n+r})$: mod p singular of p-rk r

$\implies$ Conjecture 2 is ture, i.e.,

$\exists \{S_1, \cdots, S_t\} \subset \Lambda_r^+$: fin. set s.t. level$(S_j)$ is p-power and

$$F \equiv \sum_{j=1}^{t} c_j \theta_{S_j}^{(n+r)} \bmod p$$

6. Proof of Theorem 3

6. Proof of Theorem 3

In Theorem 1, 2, we assumed $n \geq r$.

In Theorem 3, we include the case $r = 2$, $n = 1$.

Proposition 6.1

p: prime, $S \in \Lambda_2^+$: level $N = p^j \cdot N'$ with $p \nmid N'$,

$\exists \phi \in M_k(\Gamma_1)_{\mathbb{Z}_{(p)}}$ s.t. $\theta_S^{(1)} \equiv \phi \bmod p$

$$\Longrightarrow \quad N' = 1, \quad \text{i.e.,} \quad \text{level}(S) = p^j$$

Remark. To include the case $r = 2$, $n = 1$, we look at degree 1 theta!

Proof.

Suppose $N' > 1$.

Apply Theorem B (Kitaoka's original) to $\theta_S^{(1)}$ with $M = \begin{pmatrix} * & * \\ p^j & N' \end{pmatrix}$.

6. Proof of Theorem 3

➡ $\theta_S^{(1)} \mid M = \kappa \cdot \vartheta_{S'}^{(1)}$, and

$N'S'$: half integral, $(N', \mathrm{cont}(N'S')) = 1$,

$\mathrm{cont}(N'S') = p^\alpha$ if $p^\alpha \,||\, \mathrm{cont}(S)$

➡ $\frac{N'}{p^\alpha} S' \in \Lambda_2^+$: primitive

➡ $\exists l$: prime with $l \nmid N'$ s.t.

$$A\left(S', \frac{l \cdot p^\alpha}{N'}\right) = A\left(\frac{N'S'}{p^\alpha}, l\right) \not\equiv 0 \bmod p.$$

➡ ϕ: of level 1 $\Longrightarrow$ $\kappa \cdot A\left(S', \frac{l \cdot p^\alpha}{N'}\right) \equiv a_\phi\left(\frac{l \cdot p^\alpha}{N'}\right) = 0 \bmod p$

➡ This contradicts and we get $N' = 1$. □

6. Proof of Theorem 3

- Let level(S): <u>odd</u> power of p. ($\Longrightarrow$ det($2S$): odd power of p)

  By the Elementary Divisors Theorem (and by some consideration),

  we may assume that S is of the form

$$S = S(i,j) = p^i \cdot \begin{pmatrix} a & bp^{j+1} \\ bp^{j+1} & dp^{2j+1} \end{pmatrix} \quad \text{with} \quad adp - b^2p^2 = p$$

- Put $\omega(f) := \inf\{k \mid f \in \widetilde{M}_k(\Gamma_1)\}$ (filtration weight).

6. Proof of Theorem 3

Proposition 6.2

p: odd prime, $S \in \Lambda_2^+$, $\det(2S)$: odd power of p,

$$S = S(i,j) = p^i \cdot \begin{pmatrix} a & bp^{j+1} \\ bp^{j+1} & dp^{2j+1} \end{pmatrix} \quad \text{with} \quad adp - b^2p^2 = p$$

$$\Longrightarrow \quad \omega(\vartheta_S^{(1)}) \geq p^{i+j} \cdot \tfrac{p+1}{2}$$

Proof. We write $\omega(S) := \omega(\vartheta_S^{(1)})$.

Use $\vartheta_{S(i,j)}^{(1)} = \vartheta_{S(i-1,j)}^{(1)} | V(p)$. (Here $f|V(p) := f(pZ) = \sum_{n=0}^{\infty} a_f(n) \boldsymbol{e}(pnz)$)

By Serre (and Katz)
$\omega(f|V(p)) = p\omega(f|V(p))$

$\Rightarrow$ $\omega(S(i,j)) = p^i \omega(S(0,j))$.

6. Proof of Theorem 3

We can confirm $\theta^{(1)}_{S(0,j)} \mid U(p) = \theta^{(1)}_{S(1,j-1)}$.

(Here $f|U(p) := \sum_{n=0}^{\infty} a_f(pn)\boldsymbol{e}(nz)$)

$$A(S(0,j),pn) = \sharp\{(x,y) \mid ax^2 + bp^{j+1}xy + dp^{2j+1}y^2 = pn\}$$
$$= \sharp\{(px,y) \mid p^2ax^2 + bp^{j+2}xy + dp^{2j+1}y^2 = pn\}$$
$$= \sharp\{(x,y) \mid pax^2 + bp^{j+1}xy + dp^{2j}y^2 = n\} = A(S(1,j-1),n)$$

$$f|T_p \equiv f|U(p) \equiv \text{mod } p \implies \omega(f|U(p)) \leq \omega(f) \implies$$

$$\omega(S(0,j)) \geq \omega(S(1,j-1)) = p\omega(S(0,j-1)).$$

$$\omega(S(i,j)) \geq p^{i+j}\omega(S(0,0)) = p^{i+j}\omega(S(0.0)) = p^{i+j} \cdot \frac{p+1}{2}. \quad \square$$

By Boecherer-Nagaoka's result (or Serre's result for $n=1$)

$\forall S \in \Lambda_2^+$ with $\text{level}(S) = p$, $\exists f \in M_{\frac{p+1}{2}}(\Gamma_n)_{\mathbb{Z}_{(p)}}$ s.t. $f \equiv \theta^{(n)}_S \bmod p$

6. Proof of Theorem 3

Corollary 6.3

p: odd prime, $\quad S \in \Lambda_2^+$,

$\exists \phi \in M_{\frac{p+1}{2}}(\Gamma_1)_{\mathbb{Z}_{(p)}} \quad \text{s.t.} \quad \vartheta_S^{(1)} \equiv \phi \bmod p \quad \Longrightarrow \quad \text{level}(S) = p$

Proof. $\quad \vartheta_S^{(1)} \equiv \phi \bmod p \quad \Longrightarrow \quad \omega(S) = \frac{p+1}{2}$

On the other hand, $\vartheta_S^{(1)} \equiv \phi \bmod p \quad \Longrightarrow \quad \text{level}(S)$: p-power

Prop 6.1

Actually, $\det(2S)$: an odd power of p.

$\because \quad \chi_S$ has nontrivial p-component, otherwise $1 \equiv \frac{p+1}{2} \bmod p-1$.

➡ $S = S(i,j), \quad \omega(S) = \omega(S(i,j)) \geq p^{i+j} \cdot \frac{p+1}{2}$

Prop 6.2

➡ $i = j = 0$, i.e., $\text{level}(S) = p \quad \square$

We prove Theorem 3.

Theorem 3 (Boecherer-K)

$$n \geq 1, \quad r = 2, \quad p > n+5, \quad p \equiv -1 \bmod 4, \quad N = 1$$

$F \in M_{\frac{p+1}{2}}(\Gamma_{n+2})$: mod p singular of p-rk 2

$\Longrightarrow$ Conjecture 3 is ture, i.e.,

$\exists \{S_1, \cdots, S_t\} \subset \Lambda_2^+$: fin. set s.t. $\underline{\mathrm{level}(S_j) = p}$ and

$$F \equiv \sum_{j=1}^{t} c_j \theta_{S_j}^{(n+r)} \bmod p$$

Proof. $\{T_1, \cdots, T_{h_p}\} \subset \Lambda_2/GL_2(\mathbb{Z})$: level p

By Boecherer-Nagaoka's result

$$\theta_{T_i}^{(n)} \in M_1(\Gamma_0^{(n)}(p), \chi_{-p}), \quad \exists G_i \in M_{\frac{p+1}{2}}(\Gamma_n) \quad \text{s.t.} \quad G_i \equiv \theta_{T_i}^{(n)} \bmod p$$

➡ G_i: mod p singular $\because$ $\vartheta_{T_i}^{(n)}$: true singular.

Consider $$H := F - \sum_{i=1}^{h_p} \frac{1}{2} a_F\left(\begin{smallmatrix} 0 & 0 \\ 0 & T_i \end{smallmatrix}\right) G_i \in M_{\frac{p+1}{2}}(\Gamma_{n+2})$$

$A(T_i, T_i) = 2$

➡ H: mod p singular of some p-rank $r' \leq 2$.

Suppose that still $r' = 2$.

➡ $\exists S \in \Lambda_2^+$ with $a_H\left(\begin{smallmatrix} 0 & 0 \\ 0 & S \end{smallmatrix}\right) \not\equiv 0 \bmod p$ s.t. $\det S$: minimal

Cor 4.3 ➡ $\theta_S^{(1)} \in \widetilde{M}_{\frac{p+1}{2}}(\Gamma_2)$ ➡ $\text{level}(S) = p$ — Cor 6.3

6. Proof of Theorem 3

➡ $S \sim T_j \bmod \mathrm{GL}_2(\mathbb{Z})$ for some j

$$a_H\begin{pmatrix} 0 & 0 \\ 0 & T_i \end{pmatrix} \equiv 0 \bmod p \text{ for } \forall i \quad \Longrightarrow \quad a_H\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \equiv 0 \bmod p$$

Contradiction!

Therefore $r' \leq 1$ or $H \equiv 0 \bmod p$.

➡ $p + 1 - r' \equiv 0 \bmod p - 1$ Impossible!

Boecherer-K

➡ $H \equiv 0 \bmod p$ $\square$

7. Remarks and Problems

7. Remarks and Problems

How about mod p^m singular forms?

Definition

$F \in M_k(\Gamma_0^{(n+r)}(N), \chi)_{\mathbb{Z}_{(p)}}$: *mod* p^m *singular* of p-rank r

$\overset{\text{def}}{\Longleftrightarrow}$ $a_f(T) \equiv 0 \bmod p^m$ for $\forall T \in \Lambda_{n+r}$ with $\mathrm{rank}(T) \geq r$,

$\exists T \in \Lambda_{n+r}$ with $\mathrm{rank}(T) = r$ s.t. $a_F(T) \not\equiv 0 \bmod p$

We proved in 2016 that

$$F \in M_k(\Gamma_0^{(n+r)}(N), \chi)_{\mathbb{Z}_{(p)}} : \textit{mod } p^m \textit{ singular of } p\text{-rank } r,\ \chi^2 = 1$$

$$\Longrightarrow \quad 2k - r \equiv 0 \bmod (p-1)p^{m-1}$$

7. Remarks and Problems

We can prove the mod p^m version of Theorem 2

by using Theorem 2 and induction on m.

Theorem 2' (Boecherer-K)

$n \geq r, \quad p-1 \nmid r$

$F \in M_k(\Gamma_{n+r})$: mod p^m singular of p-rk r

$\exists \{S_1, \cdots, S_t\} \subset \Lambda_r^+$: fin. set s.t. level($S_j$) is p-power and

$$F \equiv \sum_{j=1}^{t} c_j \theta_{S_j}^{(n+r)} \bmod p^m$$

We want to prove mod p^m version of Theorem 3! Because ...

Relation to p-adic Siegel-Eisenstein series

$E_k^{(n)}$: Siegel-Eisenstein series of wt. k, deg. n

Nagaoka proved that

$$k_m := 1 + \frac{p-1}{2}p^{m-1} \quad \Longrightarrow$$

$$\lim_{m\to\infty} E_{k_m}^{(n)} = \text{genus theta series with level } p \ \in M_{\frac{p+1}{2}}(\Gamma_0^{(n)}(p), \chi_{-p})$$

(p-adic limit)

7. Remarks and Problems

Similarly, Katsurada-Nagaoka proved that

$$k'_m := 2 + (p-1)p^{m-1} \quad \Longrightarrow$$

$$\lim_{m\to\infty} E^{(n)}_{k'_m} = \text{genus theta series with level } p \ \in M_{p-1}(\Gamma_0^{(n)}(p))$$

(p-adic limit)

They proved these formulas by caluculation of local densities.

Since we have

$E^{(n)}_{k_m}$: mod $p^{c(m)}$ singular of p-rk 2

$E^{(n)}_{k'_m}$: mod $p^{c'(m)}$ singular of p-rk 4

(by FC of Eisen)

we expect to give alternative proofs in terms of mod p^m singular

Thank you for your attention!