Square-free レベルの 2 次 Siegel カスプ形式に対する Rankin-Selberg 積分について

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Introduction

L-functions for Siegel cusp forms (degree 2)

- $\mathfrak{h}_2 = \left\{ Z \in \mathrm{M}_2(\mathbb{C}) \;\middle|\; {}^t Z = Z, \mathrm{Im}(Z) > 0 \right\}$.

A Siegel cusp form F of weight $\ell \in \mathbb{Z}_{>0}$ for $\mathrm{Sp}_2(\mathbb{Z})$ has the Fourier expansion:

$$F(Z) = \sum_{T \in \mathcal{Q}^+} a_F(T) \exp(2\pi i \cdot \mathrm{tr}(TZ)) \quad (Z \in \mathfrak{h}_2)$$

where $\mathcal{Q}^+ = \left\{ \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} > 0 \;\middle|\; a, b, c \in \mathbb{Z} \right\}$.

Assume that F is an eigenfunction of all Hecke operators.
$T \in \mathcal{Q}^+$: fixed
Andrianov(1971) suggested the following Dirichlet series

$$L(s,T,F) := \sum_{m=1}^{\infty} \frac{a_F(mT)}{m^s} \qquad (\mathrm{Re}(s) \gg 0).$$

$L(s,T,F)$ has

- Integral representation
  (Andrianov's integral (The Rankin-Selberg integral))
- Meromorphic continuation to $\mathbb{C}$
- Euler product

Today's talk
We consider the Rankin-Selberg integral for Siegel cusp forms for square-free level.

Motivation

By a consideration of a trace formula, we have

$$\sum_{F\in\mathcal{B}(l,N)} (\text{The RS integral of } F)\cdot(\text{Bessel period of } F)$$

$$=\sum(\text{Orbital integral}).$$

($\mathcal{B}(l,N)$:an orthonormal basis of the space of SCFs of wt. l and level N.)

It can be expected that this formula implies an asymptotic formula (as N grows to $+\infty$) of a certain weighted average of the spinor L-functions for Siegel cusp forms.

$\longrightarrow$

- A non-vanishing property of special L-values.
- A weighted equidistribution theorem of the Satake parameters of SCFs.

Preceeding study

- Tsuzuki (2020) : Weight aspect ($l\to+\infty$, $N=1$.)

Today's topics

- Bessel model
- Local zeta integral
- The Rankin-Selberg integral
- Asymptotic formula(Future work)

Notations

Let

$$\mathrm{G} = \mathrm{GSp}_2 = \left\{ g \in \mathrm{GL}_4 \;\middle|\; {}^t g \begin{bmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{bmatrix} g = \nu(g) \begin{bmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{bmatrix} \right\}.$$

$\nu : \mathrm{G} \to \mathrm{GL}_1$: similitude norm.

Let $\mathrm{B} = \mathrm{MN}$ be the Siegel parabolic subgroup of G where

$$\mathrm{M} = \left\{ \begin{bmatrix} A & 0 \\ 0 & {}^t A^{-1} c \end{bmatrix} \;\middle|\; c \in \mathrm{GL}_1, A \in \mathrm{GL}_2 \right\},$$
$$\mathrm{N} = \left\{ \begin{bmatrix} \mathbf{1}_2 & X \\ 0 & \mathbf{1}_2 \end{bmatrix} \;\middle|\; X \in \mathrm{M}_2, X = {}^t X \right\}.$$

- For a prime number p, and a non-zero ideal $\mathfrak{a} \subset \mathbb{Z}_p$,

$$\mathbf{K}_p = \mathrm{G}(\mathbb{Z}_p), \quad \mathbf{K}_0(\mathfrak{a}) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbf{K}_p \;\middle|\; C \in \mathfrak{a}\mathrm{M}_2(\mathbb{Z}_p) \right\}.$$

- $\mathbf{K}_0(N) = \prod_{p<\infty} \mathbf{K}_0(N\mathbb{Z}_p) \quad (N \in \mathbb{N})$.
- $\mathbf{K}_\infty = \mathrm{G}(\mathbb{R})^0 \cap \mathrm{O}(4)$.
- $\mathbf{K} = \prod_{p\leq\infty} \mathbf{K}_p$

Local Bessel models for G

Bessel models for G are introduced as a substitute for Whittaker models.
The uniqueness theorem of Bessel models holds.
(Novodvorski&Piatetski-Shapiro(1973), Prasad&Takloo-Bighash(2011)).

The local Bessel models

- F : a non-archimedean local field with $\mathrm{char}(F) = 0$.
- $\mathfrak{o} \subset F$: the ring of integers.
- $\mathfrak{p} \subset \mathfrak{o}$: the maximal ideal.
- $\varpi \in \mathfrak{p}$: a prime element.
- $q := \#(\mathfrak{o}/\mathfrak{p})$
- $\psi : F \to \mathbb{C}^1$: a non-trivial character.

For a fixed symmetric matrix $S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \in \mathrm{Sym}_2(F)$ with $b^2 - 4ac \neq 0$,

$$L_S := F\mathbf{1}_2 + F\xi \quad (\xi = \begin{bmatrix} b/2 & c \\ -a & -b/2 \end{bmatrix}), \quad T_S := L_S^\times.$$

$\longrightarrow T_S = \{g \in \mathrm{GL}_2(F) \mid {}^t g S g = \det g \cdot S\}.$

Bessel subgroup

Then the Bessel subgroup of $\mathrm{G}(F)$ w.r.t. S is defined by

$$R_S := \left\{ \begin{bmatrix} g & \\ & \det(g)\cdot {}^t g^{-1} \end{bmatrix} n \;\middle|\; g \in T_S, n \in \mathrm{N}(F) \right\}.$$

Let Λ be a character on T_S and ψ_S be the character on $\mathrm{N}(F)$ given by

$$\psi_S(\begin{bmatrix} \mathbf{1}_2 & X \\ & \mathbf{1}_2 \end{bmatrix}) = \psi(\mathrm{tr}(SX)).$$

Then, $\Lambda \otimes \psi_S : R_S \ni \begin{bmatrix} g & \\ & \det(g)\cdot {}^t g^{-1} \end{bmatrix} n \mapsto \Lambda(g)\psi_S(n) \in \mathbb{C}^1$ defines a character on R_S.

For a smooth adm. irred. rep'n π of $\mathrm{G}(F)$, let $(\pi^*)^{S,\Lambda}$ be the space of all linear functional $\ell : \pi \to \mathbb{C}$ s.t. $\ell(\pi(r)v) = \Lambda \otimes \psi_S(r)\ell(v)$ for all $r \in R_S$, $v \in \pi$.

Uniqueness theorem

Let π be a preunitary smooth adm. irred. rep'n of $\mathrm{G}(F)$. Then,

$$\dim(\pi^*)^{S,\Lambda} \leq 1.$$

π admits a local (S, Λ)-Bessel model $\overset{\mathrm{def}}{\Longleftrightarrow}$ $(\pi^*)^{S,\Lambda} \neq \{0\}$

If π admits a local (S,Λ)-Bessel model, we choose $\ell \in (\pi^*)^{S,\Lambda} - \{0\}$ and define

$$\mathscr{B}(S,\Lambda)[\pi] := \{B : \mathrm{G}(F) \to \mathbb{C} \mid \exists \xi \in \pi, \forall g \in \mathrm{G}(F), B(g) = \ell(\pi(g)\xi)\}.$$

This space is called the local (S,Λ)-Bessel model.

Assumptions

- π is of type I, IIb, IIIa or VIb (in the sense of Schmidt&Tran's classification)
- L_S/F is an unramified field extension.

(cf.) R. Schmidt, L. Tran, *Zeta integrals for GSp_4 via Bessel models*, Pacific J. Math, **296** (2018), 437–480.

Subgroup $\mathrm{G}^{\#}(F) \leq \mathrm{G}(F)$

Let $\mathrm{G}^{\#}(F) := \{g \in \mathrm{GL}_2(L_S) \mid \det(g) \in F^{\times}\}$. We consider the symplectic form on $L_S^2 \cong F^4$:

$$\rho(x, y) := \det(\left[\begin{smallmatrix} x_1 & y_1 \\ x_2 & y_2 \end{smallmatrix}\right]), \quad x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right], y = \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] \in L_S^2.$$

Then $\rho(gx, gy) = \det(g) \cdot \rho(x, y)$ for $\forall x, y \in L_S^2$, $\forall g \in \mathrm{G}^{\#}(F)$. Hence we get an embedding $\iota : \mathrm{G}^{\#}(F) \hookrightarrow \mathrm{G}(F)$.

Let

- $\mathbf{K} = \mathrm{G}(\mathfrak{o})$.
- $\mathbf{K}_0(\mathfrak{p}) = \left\{ \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathbf{K}_F \;\middle|\; C \in \mathfrak{p}M_2(\mathfrak{o}) \right\}$.
- $\mathbf{K}^{\#} := \iota^{-1}(\mathbf{K})$.

- π : a smooth adm. irred. rep'n of $\mathrm{G}(F)$ which admits a local (S,Λ)-Bessel model.

For $\mathrm{Re}(s)>1$ and $B\in\mathscr{B}(S,\Lambda)[\pi]^{\mathbf{K}_0(\mathfrak{p})}$, let

$$Z(s,B)=\int_{F^\times}\int_{\mathbf{K}^\#}B\left(\begin{bmatrix} a\mathbf{1}_2 & \\ & \mathbf{1}_2\end{bmatrix}\iota(k^\#)\right)|a|^{s-1}dk^\# d^\times a.$$

Let us define operators $\eta, T_0\in\mathrm{End}\left(\mathscr{B}(S,\Lambda)[\pi]^{\mathbf{K}_0(\mathfrak{p})}\right)$ by

$$[\eta B](g)=B(g\eta)\quad\left(\eta=\begin{bmatrix} & & & -1\\ & & 1 & \\ & \varpi & & \\ -\varpi & & & \end{bmatrix}:\text{Atkin-Lehner element}\right),$$

$$[T_0B](g)=\frac{1}{\mathrm{vol}(\mathbf{K}_0(\mathfrak{p});dh)}\int_{\mathbf{K}_0(\mathfrak{p})\begin{bmatrix}\varpi\mathbf{1}_2 & \\ & \mathbf{1}_2\end{bmatrix}\mathbf{K}_0(\mathfrak{p})}B(gh)dh.$$

Theorem(K. Tsuzuki)

Suppose $\Lambda = \mathbf{1}$ and π is of type I, IIb, IIIa or VIb.
For $B \in \mathscr{B}(S, \mathbf{1})[\pi]^{\mathbf{K}_0(\mathfrak{p})}$, the following formulas hold.

- Local zeta integral for newforms : If π is of type IIIa or VIb, we have

$$Z(s, B) = \frac{q^{s+1}}{q^2+1} \cdot \frac{L(s+\frac{1}{2}, \pi)}{\zeta_{L_S}(s+1)} \cdot [\eta B](\mathbf{1}_4).$$

- Local zeta integral for old forms : If π is of type I or IIb, we have

$$\begin{aligned} Z(s, B) = & \frac{1}{q^2+1} \cdot \frac{L(s+\frac{1}{2}, \pi)}{\zeta_{L_S}(s+1)} \\ & \times \left\{ B + q^{-1} T_0 \eta B + q^{s+1}(1 - \mu(\pi) q^{-s} + q^{-2s}) \eta B \right\} (\mathbf{1}_4). \end{aligned}$$

where $\mu(\pi) = q^{-\frac{1}{2}}(\alpha_\pi + \beta_\pi + \alpha_\pi^{-1} + \beta_\pi^{-1})$, (α_π, β_π) is the Satake parameter of π, and $L(s, \pi)$ is the spinor L-function of π

The spinor L-function

(cf.) I.I. Piatetski-Shapiro, *L-functions for GSp_4*, Pacific J. Math. Special Issue (1997), 259–275.

- π is of type I or IIb
  $\longrightarrow L(s,\pi) = \{(1-\alpha_\pi q^{-s})(1-\beta_\pi q^{-s})(1-\alpha_\pi^{-1} q^{-s})(1-\beta_\pi^{-1} q^{-s})\}^{-1}$
  where (α_π, β_π) is the Satake parameter of π_p.
- π is of type IIIa or VIb
  $\longrightarrow$ Schmidt&Tran(2018)

(cf.) R. Schmidt, L. Tran, *Zeta integrals for GSp_4 via Bessel models*, Pacific J. Math, **296** (2018), 437–480.

Remark

- π is of type I or IIb
- $B \in \mathscr{B}(S,\Lambda)[\pi]^{\mathbf{K}}$

$$\Longrightarrow Z(s,B) = \frac{L(s+\frac{1}{2},\pi)}{L(s+1,\Lambda)} B(\mathbf{1}_2).$$

Most of parts of the proof are direct calculations.

Fact

The classification of all irreducible admissible Iwahori spherical representations of $\mathrm{G}(F)$ which admits a local Bessel model is given by Pitale and Schmidt.

(cf.) A. Pitale, R. Schmidt, *Bessel models for GSp_4 : Siegel vectors of square-free level*, J. Number Theory, $\mathbf{136}$ (2014), 134–164.

Tools

- Calculations for newforms
  Many recurrence relations of local Bessel models are given by Pitale & Schmidt(2014).
- Calculations for old forms
  Local functional equation for $L(s,\pi)$ (Piatetski-Shapiro(1997)).

Outline of proof

By a left coset decomposition

$$\mathbf{K}^{\#} = \mathbf{K}_0^{\#}(\mathfrak{p}) \bigsqcup \left(\bigsqcup_{x \in \mathfrak{o}/\mathfrak{p}} \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right] \mathbf{K}_0^{\#}(\mathfrak{p}) \right)$$

and a direct calculation, we have

$$Z(s,B) = \frac{1}{q^2+1} \sum_{n \geq 0} \left\{ B\left(\left[\begin{smallmatrix} \varpi^n \mathbf{1}_2 & \\ & \mathbf{1}_2 \end{smallmatrix}\right]\right) + q^{s+1} \cdot \eta B\left(\left[\begin{smallmatrix} \varpi^n \mathbf{1}_2 & \\ & \mathbf{1}_2 \end{smallmatrix}\right]\right) \right\} q^{-n(s-1)}$$

Outline for type IIIa, VIb (newforms)

- $\exists \{B_i\}_{1 \leq i \leq \dim(\mathscr{B}(S,\mathbf{1})[\pi]^{\mathbf{K}_0(\mathfrak{p})})}$: Eigenbasis of T_0.
- $\left\{B_i\left(\left[\begin{smallmatrix} \varpi^n \mathbf{1}_2 & \\ & \mathbf{1}_2 \end{smallmatrix}\right]\right)\right\}_{n \in \mathbb{Z}_{\geq 0}}$ and $\left\{\eta B_i\left(\left[\begin{smallmatrix} \varpi^n \mathbf{1}_2 & \\ & \mathbf{1}_2 \end{smallmatrix}\right]\right)\right\}_{n \in \mathbb{Z}_{\geq 0}}$ are geometric series (Pitale, Schmidt (2014)).
- $B_i(\mathbf{1}_4)$ and $\eta B_i(\mathbf{1}_4)$ are determined by means of recurrence relations of local Bessel models investigated by Pitale&Schmidt.

Outline for old forms

Assume that π is of type I.

It is known that $= \dim \mathscr{B}(S, \mathbf{1})[\pi]^{\mathbf{K}_0(\mathfrak{p})} = 4$.

$\{B_i\}_{1 \le i \le 4}$: Eigenbasis of T_0. Let $T_0 B_i = \lambda_i B_i$ and $\eta B_i = \sum_{i=1}^{4} \eta_{ji} B_j$.

It is known that $B_i \left(\begin{bmatrix} \varpi^{n+1}\mathbf{1}_2 & \\ & \mathbf{1}_2 \end{bmatrix} \right) \overset{\text{Pitale Schmidt}}{=} \lambda_i q^{-3} B_i \left(\begin{bmatrix} \varpi^{n}\mathbf{1}_2 & \\ & \mathbf{1}_2 \end{bmatrix} \right) \quad (n \ge 0)$.

$$\begin{aligned}
&Z(s, B_i) \\
&= \frac{1}{q^2+1} \sum_{n=0}^{\infty} \left\{ B_i \left(\begin{bmatrix} \varpi^{n}\mathbf{1}_2 & \\ & \mathbf{1}_2 \end{bmatrix} \right) + \frac{q^2}{X} \cdot \eta_p B_i \left(\begin{bmatrix} \varpi^{n}\mathbf{1}_2 & \\ & \mathbf{1}_2 \end{bmatrix} \right) \right\} X^n \ (X = q^{-s+1}) \\
&= \frac{1}{q^2+1} \left\{ \frac{B_i(\mathbf{1}_4)}{1 - \lambda_i q^{-3} X} + \frac{q^2}{X} \cdot \sum_{j=1}^{4} \frac{\eta_{ji} B_j(\mathbf{1}_4)}{1 - \lambda_j q^{-3} X} \right\} \\
&= \frac{1}{q^2+1} \cdot \frac{1}{X Q(X)} \left(A_0 + A_1 X + A_2 X^2 + A_3 X^3 + A_4 X^4 \right) \\
&\qquad \left(Q(X) = \prod_{i=1}^{4} \frac{1}{1 - \lambda_i q^{-3} X} \right).
\end{aligned}$$

By direct calculation, we have

$$\cdot\ A_0 = q^2 \sum_{j=1}^{4} \eta_{ji} B_j(\mathbf{1}_4) = q^2 \eta B_i(\mathbf{1}_4).$$

$$\begin{aligned}\cdot\ A_1 &= B_i(\mathbf{1}_4) + q^2 \sum_{j=1}^{4} \eta_{ji} \left(q^{-3}\lambda_j - q^{-3} \sum_{i=1}^{4} \lambda_i \right) \cdot B_j(\mathbf{1}_4) \\ &= \left\{ B_i + q^{-1} T_0 \eta B_i - q\lambda(\pi)\eta B_i \right\} (\mathbf{1}_4). \\ &\left(\because \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \left\{ q^{\frac{3}{2}}\alpha_\pi, q^{\frac{3}{2}}\beta_\pi, q^{\frac{3}{2}}\alpha_\pi^{-1}, q^{\frac{3}{2}}\beta_\pi^{-1} \right\} \right)\end{aligned}$$

$$
\begin{aligned}
&\frac{\zeta_{L_S}(s+1)}{L(s+\frac{1}{2},\pi)} Z(s,B_i) \\
&= \frac{Q(X)}{1-q^{-4}X^2} \cdot Z(s,B_i) \\
&= \frac{1}{q^2+1} \cdot \frac{A_0 + A_1X + A_2X^2 + A_3X^3 + A_4X^4}{(1-q^{-4}X^2)X}.
\end{aligned}
$$

By the local functional equation (investigated by Piatetski-Shapiro), the above rational function is invariant under $X \longleftrightarrow q^2X^{-1}$.
Hence, $A_2 = q^{-4}(q^2-1)A_0$, $A_3 = -q^{-4}A_1$, $A_4 = -q^{-6}A_0$.

$$
\begin{aligned}
\therefore &\frac{\zeta_{L_S}(s+1)}{L(s+\frac{1}{2},\pi)} Z(s,B_i) \\
&= \frac{1}{q^2+1} \cdot X^{-1}(A_0 + A_1X + q^{-2}A_0X^2) \\
&= \frac{1}{q^2+1} \cdot \left\{ B_i + q^{-1}T_0(\eta B_i) + p^{s+1}\left(1-\mu(\pi)q^{-s}+q^{-2s}\right)\eta B_i \right\}(\mathbf{1}_4).
\end{aligned}
$$

$\square$

Global Bessel models for Siegel cusp forms

Let

- $l \in \mathbb{Z}_{\geq 0}$, $N \in \mathbb{Z}_{>0}$.
- $\psi = \prod_{p \leq \infty} \psi_p$: the standard non-trivial character on $\mathbb{A}/\mathbb{Q}$

$\Pi_{\mathrm{cusp}}(l, N)$: The set of all irreducible cuspidal representations $\pi \cong \otimes'_{p \leq \infty} \pi_p$ of $\mathrm{Z}(\mathbb{A})\backslash\mathrm{G}(\mathbb{A})$ having the following properties:

(i) As a $(\mathfrak{g}, \mathbf{K}_\infty)$-module, $\pi_\infty \cong D_l^+ \oplus D_l^-$ where D_l^+ (resp. D_l^-) is the holomorphic (resp. anti-holomorphic) discrete series representation of $\mathrm{Sp}_2(\mathbb{R})$ of weight l.

(ii) $\pi^{\mathbf{K}_0(N)} \neq \{0\}$.

For a symmetric matrix $S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \in \mathrm{Sym}_2(\mathbb{Q})$ with $b^2 - 4ac \neq 0$, we define L_S, T_S, R_S, and ψ_S as above.

- $\pi \cong \otimes'_{p\leq\infty}\pi_p \in \Pi_{\mathrm{cusp}}(l,N)$.
- $\mathbb{A}_{L_S} := L_S \otimes_{\mathbb{Q}} \mathbb{A}$.
- $\Lambda = \prod_{p\leq\infty}\Lambda_p$: a character on $\mathbb{A}_{L_S}^{\times}/L_S^{\times}$.

The (S,Λ)-Bessel period : For $\varphi \in \pi$, $g \in \mathrm{G}(\mathbb{A})$,

$$B^{S,\Lambda}(\varphi;g) := \int_{\mathrm{Z}(\mathbb{A})R_S(\mathbb{Q})\backslash R_S(\mathbb{A})} \Lambda\otimes\psi_S(r)^{-1}\varphi(rg)dr.$$

π admits a global (S,Λ)-Bessel model $\overset{\text{def}}{\Longleftrightarrow} \exists\varphi\in\pi$, $\exists g\in\mathrm{G}(\mathbb{A})$, $B^{S,\Lambda}(\varphi;g)\neq 0$.

Remark

If π admits a global (S,Λ)-Bessel model, we have

- π_p admits a local (S,Λ_p)-Bessel model.
- $\Lambda|_{\mathbb{A}^\times} = \omega_\pi$.
- L_S is an imaginary quadratic field.

Let

- $D < 0$: a fundamental discriminant.
- $E = \mathbb{Q}(\sqrt{D})$.
- $\theta \in \mathfrak{o}_E$ s.t. $\theta - \overline{\theta} = -\sqrt{D}$.
- $T_\theta := \begin{bmatrix} 1 & \mathrm{tr}_{E/\mathbb{Q}}(\theta)/2 \\ \mathrm{tr}_{E/\mathbb{Q}}(\theta)/2 & \mathrm{N}_{E/\mathbb{Q}}(\theta) \end{bmatrix}$.
- ω : a character on $\mathrm{Cl}(E) \cong \mathbb{A}_E^\times / E^\times E_\infty^\times \widehat{\mathfrak{o}_E}^\times$,

Then $L_{T_\theta} \cong E$.

$\Pi_{\mathrm{cusp}}^{(T_\theta,\omega)}(l, N)$: The set of all $\pi \in \Pi_{\mathrm{cusp}}(l, N)$ admitting a global (T_θ, ω)-Bessel model.

Fact

For $\pi \cong \otimes'_{p \le \infty} \pi_p \in \Pi_{\mathrm{cusp}}^{(T_\theta,\omega)}(l, N)$, we have

- $p < \infty$, $\pi_p^{\mathbf{K}_p} \neq \{0\} \Longrightarrow \pi_p$ is of type I or IIb
- $p < \infty$, $\pi_p^{\mathbf{K}_p} = \{0\} \Longrightarrow \pi_p$ is of type IIIa or VIb

The Rankin-Selberg integral for Siegel cusp forms

Subgroup $\mathrm{G}^{\#}$

Let $\mathrm{G}^{\#}(\mathbb{Q}) := \{g \in \mathrm{GL}_2(E) \mid \det(g) \in \mathbb{Q}^{\times}\}$. We consider the symplectic form on $E^2 \cong \mathbb{Q}^4$:

$$\rho(x, y) := \det(\tfrac{1}{\sqrt{D}} \left[\begin{smallmatrix} x_1 & y_1 \\ x_2 & y_2 \end{smallmatrix}\right]), \quad x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right], y = \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] \in E^2.$$

Then $\rho(gx, gy) = \det(g) \cdot \rho(x, y)$ for $\forall x, y \in E^2$, $\forall g \in \mathrm{G}^{\#}(\mathbb{Q})$. Hence we get an embedding $\iota_\theta : \mathrm{G}^{\#} \hookrightarrow \mathrm{G}$ as algebraic groups over $\mathbb{Q}$.

Let $\mathrm{B}^{\#} := \iota_\theta^{-1}(\mathrm{B})$, $\mathbf{K}_p^{\#} := \iota_\theta^{-1}(\mathbf{K}_p)$, and $\mathbf{K}^{\#} := \prod_{p \le \infty} \mathbf{K}_p^{\#}$.

Eisenstein series on $\mathrm{G}^{\#}(\mathbb{A})$

Let $s \in \mathbb{C}$ and $\omega \in \widehat{\mathrm{Cl}(E)}$. By Iwasawa decomposition $\mathrm{G}^{\#}(\mathbb{A}) = \mathrm{B}^{\#}(\mathbb{A})\mathbf{K}^{\#}$, we define a function $f^{(s,\omega)} : \mathrm{G}^{\#}(\mathbb{A}) \to \mathbb{C}$ as

$$f^{(s,\omega)}\left(\begin{bmatrix} a\tau & \beta \\ 0 & \tau^{-1} \end{bmatrix} k^{\#}\right) = \omega(\tau)^{-1}|\mathrm{N}_{E/\mathbb{Q}}(\tau)|_{\mathbb{A}}^{s+1}|a|_{\mathbb{A}}^{s+1} \quad (a \in \mathbb{A}^{\times},\ \tau \in \mathbb{A}_E^{\times},\ k^{\#} \in \mathbf{K}^{\#}).$$

The Eisenstein series:

$$E(s,\omega,g^{\#}) = \sum_{\gamma^{\#} \in \mathrm{B}^{\#}(\mathbb{Q})\backslash \mathrm{G}^{\#}(\mathbb{Q})} f^{(s,\omega)}(\gamma^{\#} g^{\#})$$

converges absolutely for $\mathrm{Re}(s) > 1$.

Standard properties :

- Meromorphic continuation to $\mathbb{C}$.
- Functional equation : $E^{*}(s,\omega,g^{\#}) = E^{*}(-s,\omega^{-1},g^{\#})$ where

$$E^{*}(s,\omega,g^{\#}) = |D|^{\frac{s+1}{2}}\widehat{L}(s+1,\omega^{-1})E(s,\omega,g^{\#}).$$

Siegel cusp forms

Let

- $\mathfrak{h}_2 := \{X + iY \mid X, Y \in \mathrm{Sym}_2(\mathbb{R}), Y > 0\}$.
- $g\langle Z\rangle = (AZ+B)(CZ+D)^{-1} \quad (g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{G}(\mathbb{R})^0, Z \in \mathfrak{h}_2)$.
- $J(g, Z) = \nu(g)^{-1}\det(CZ+D)$.

Let $S_l(\mathbf{K}_0(N))$ be the space of all smooth functions $\varphi : \mathrm{G}(\mathbb{A}) \to \mathbb{C}$ s.t.

(i) $\varphi(z\gamma g) = \varphi(g)$ for $(z, \gamma, g) \in \mathrm{Z}(\mathbb{A}) \times \mathrm{G}(\mathbb{Q}) \times \mathrm{G}(\mathbb{A})$.

(ii) $\varphi(gk_{\mathrm{fin}}k_\infty) = \tau_l(k_\infty)^{-1}\varphi(g)$ for $(g, k_{\mathrm{fin}}, k_\infty) \in \mathrm{G}(\mathbb{A}) \times \mathbf{K}_0(N) \times \mathbf{K}_\infty$ where $\tau_l(k_\infty) = J(k_\infty, i\mathbf{1}_2)^l$.

(iii) $R(X)\varphi = 0$ for $X \in \mathfrak{p}^-$.

(iv) φ is bounded.

The classical Siegel cusp form F_φ corresponding to $\varphi \in S_l(\mathbf{K}_0(N))$ is defined by

$$F_\varphi(Z) = J(g_\infty, i\mathbf{1}_2)^l\varphi(g_\infty) \qquad (Z \in \mathfrak{h}_2)$$

where $g_\infty \in \mathrm{G}(\mathbb{R})^0$ s.t. $g_\infty\langle i\mathbf{1}_2\rangle = Z$.

The Rankin-Selberg integral (a reformation of Andrianov's integral)

For $\varphi \in S_l(\mathbf{K}_0(N))$, the Rankin-Selberg integral is defined by

$$\left\langle E(s,\omega), R(b_{\mathbb{R}}^{\theta})\varphi \right\rangle = \int_{\mathrm{Z}^{\#}(\mathbb{A})\mathrm{G}^{\#}(\mathbb{Q})\backslash \mathrm{G}^{\#}(\mathbb{A})} E(s,\omega,g^{\#})\varphi(\iota_{\theta}(g^{\#})b_{\mathbb{R}}^{\theta})dg^{\#}$$

where

- $b_{\mathbb{R}}^{\theta} := \begin{bmatrix} 1 & 2^{-1}\mathrm{tr}_{E/\mathbb{Q}}(\theta) & 0 & 0 \\ 0 & -2^{-1}\sqrt{|D|} & 0 & 0 \\ 0 & 0 & 2^{-1}\sqrt{|D|} & 2^{-1}\mathrm{tr}_{E/\mathbb{Q}}(\theta) \\ 0 & 0 & 0 & -1 \end{bmatrix}^{-1} \in \mathrm{G}(\mathbb{R})$,

The integral converges absolutely for $s \in \mathbb{C}$ where $E(s,\omega,g^{\#})$ is regular.

Main result

Wighted average of Fourier coefficients ($R(\varphi, E, \omega)$)

- $\mathcal{Q}^+_{\mathrm{prim}}(D) := \left\{ \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \mid a > 0,\ b^2 - 4ac = D,\ \mathrm{GCD}(a,b,c) = 1 \right\}$
- $\mathcal{Q}^+_{\mathrm{prim}}(D) \curvearrowleft \mathrm{SL}_2(\mathbb{Z}) : T \cdot \gamma := {}^t\gamma T \gamma$

There exists a natural bijection

$$\mathcal{Q}^+_{\mathrm{prim}}(D)/\mathrm{SL}_2(\mathbb{Z}) \cong \mathrm{Cl}(E)$$

For $\varphi \in S_l(\mathbf{K}_0(N))$, F_φ has the Fourier expansion of the form

$$F_\varphi(Z) = \sum_{T \in \mathcal{Q}^+} a_\varphi(T) \exp(2\pi i \mathrm{tr}(TZ)).$$

For a character $\omega : \mathrm{Cl}(E) \to \mathbb{C}^\times$, define

$$R(\varphi, E, \omega) = \frac{1}{w_D} \sum_{[T] \in \mathcal{Q}^+_{\mathrm{prim}}(D)/\mathrm{SL}_2(\mathbb{Z})} a_\varphi(T) \omega([T]).$$

where $w_D = \#\mathfrak{o}_E^\times$.

Bessel data

Let $\pi \cong \otimes'_{p\le\infty}\pi_p \in \Pi^{(T_\theta,\omega)}_{\rm cusp}(l,N)$.

For $p<\infty$ s.t. π_p is $\mathbf{K}_p$-spherical, there exist $\ell^0_{\pi_p}\in(\pi_p^*)^{T_\theta,\omega_p}$ and $\xi^0_{\pi_p}\in\pi_p^{\mathbf{K}_p}$ s.t.

$$\ell^0_{\pi_p}(\xi^0_{\pi_p})=1,\ (\xi^0_{\pi_p}|\xi^0_{\pi_p})_{\pi_p}=1.$$

A family of pairs $\{(\ell_p,\xi_p)\}_{p<\infty}\in\prod_{p<\infty}\left\{(\pi_p^*)^{T_\theta,\omega}\times\pi_p^{\mathbf{K}_0(N\mathbb{Z}_p)}\right\}$ is said to be a (T_θ,ω)-Bassel data for π if $(\ell_p,\xi_p)=(\ell^0_{\pi_p},\xi^0_{\pi_p})$ for $p\nmid N$ and $\ell_p(\xi_p)=1$ for all p.

We fix a (T_θ,ω)-Bassel data $\{(\ell_p,\xi_p)\}_{p<\infty}$ and put $\varphi^0_\pi:=v_l\otimes(\otimes_{p<\infty}\xi_p)\in\pi^{\mathbf{K}_0(N)}$. ($v_l\in D_l^+$: a lowest weight vector)

Assumptions

- N is square-free
- All primes p dividing N are inert in $E/\mathbb{Q}$.

Basic identity(Piatetski-Shapiro)

For $\pi \in \Pi_{\mathrm{cusp}}(l, N)$, $\varphi = v_\ell \otimes (\otimes_{p<\infty} \phi_p) \in \pi^{\mathbf{K}_0(N)}$.

$$\begin{aligned}
&\left\langle E(s,\omega), R(b_{\mathbb{R}}^{\theta})\varphi \right\rangle \\
&= \frac{\sqrt{|D|}}{2} \int_{\mathbb{A}^\times} \int_{\mathrm{K}^\#} B^{T_\theta,\omega}\left(\varphi; \left[\begin{smallmatrix} a\mathbf{1}_2 & \\ & \mathbf{1}_2 \end{smallmatrix}\right] \iota_\theta(k^\#) b_{\mathbb{R}}^{\theta}\right) |a|_{\mathbb{A}}^{s-1} dk^\# d^\times a \\
&= \frac{\sqrt{|D|}}{2} \times Z_\infty(\varphi, s, \omega) \times \prod_{p<\infty} Z_p(\phi_p, s, \omega_p)
\end{aligned}$$

where

- $Z_\infty(\varphi, s, \omega) = R(\varphi, E, \omega) \cdot \int_0^\infty B_{\infty,l}^{T_\theta}\left(\left[\begin{smallmatrix} a\mathbf{1}_2 & \\ & \mathbf{1}_2 \end{smallmatrix}\right] b_{\mathbb{R}}^{\theta}\right) a^{s-1} d^\times a.$
- $B_{\infty,l}^{T_\theta}(g_\infty) = J(g_\infty, i\mathbf{1}_2)^{-l} \exp(2\pi i \mathrm{tr}(T g_\infty \langle i\mathbf{1}_2 \rangle)) \quad (g_\infty \in \mathrm{G}(\mathbb{R})^0).$
- $Z_p(\phi_p, s, \omega_p) = \int_{\mathbb{Q}_p^\times} \int_{\mathrm{K}_p^\#} \ell_p(\pi_p(\left[\begin{smallmatrix} a\mathbf{1}_2 & \\ & \mathbf{1}_2 \end{smallmatrix}\right] \iota_\theta(k^\#))\phi_p) |a|_p^{s-1} dk^\# d^\times a.$

The global L-function

For $\pi \cong \otimes'_{p\le\infty} \pi_p \in \Pi_{\mathrm{cusp}}(\ell, N)$,

$$L(s,\pi) := \prod_{p<\infty} L(s, \pi_p) \quad (\mathrm{Re}(s) > \tfrac{5}{2}).$$

$$\widehat{L}(s,\pi) := \Gamma_{\mathbb{C}}(s + \tfrac{1}{2}) \Gamma_{\mathbb{C}}(s + \ell - \tfrac{3}{2}) \times L(s,\pi).$$

Theorem(K. &Tsuzuki)

Let $\pi \in \Pi_{\mathrm{cusp}}^{(T_\theta,\omega)}(l,N)$, $\varphi = v_l \otimes (\otimes_{v<\infty}\phi_p) \in \pi^{\mathbf{K}_0(N)}$.
Assume that

- $R(\eta_p)\varphi = \sigma_p\varphi \quad (\eta_p \in G(\mathbb{Q}_p)$: Atkin-Lehner element, $p|N, \sigma_p = \pm 1)$.

Then,

$$\langle E(s,\omega), R(b_{\mathbb{R}}^\theta)\varphi\rangle = 2^{-1}(-1)^l|D|^{\frac{-s-l+2}{2}}\sigma_\pi N_\pi^{s+1}\prod_{p|N_\pi}(p^2+1)^{-1}$$
$$\times R(\varphi_\pi^0, E,\omega)\frac{\widehat{L}(s+\frac{1}{2},\pi)}{\widehat{L}(s+1,\omega^{-1})}\prod_{p|N_\pi}\ell_p(\phi_p)\prod_{p|\frac{N}{N_\pi}}Z_p^*(\phi_p;s,\omega_p),$$

where N_π is the conductor of π, $\sigma_\pi = \prod_{p|N_\pi}\sigma_p$ and

$$Z_p^*(\phi_p;s,\omega_p) = \frac{L(s+1,\omega_p^{-1})}{L(s+\frac{1}{2},\pi_p)}Z_p(\phi_p;s,\omega_p).$$

In particular, if $N_\pi = N$,

$$\langle E(s,\omega), R(b_{\mathbb{R}}^\theta)\varphi\rangle = 2^{-1}(-1)^l|D|^{\frac{-s-l+2}{2}}\sigma_\pi N^{s+1}\prod_{p|N}(p^2+1)^{-1}\cdot R(\varphi,E,\omega)\frac{\widehat{L}(s+\frac{1}{2},\pi)}{\widehat{L}(s+1,\omega^{-1})}.$$

Corollary(functional equation)

Let $\pi \in \Pi_{\text{cusp}}^{(T_\theta,\omega)}(l,N)^{\text{new}}$. Assume that there exist $\varphi \in S_l(\mathbf{K}_0(N)) \cap \pi$, an imaginary quadratic field E , and $\omega \in \widehat{\mathrm{Cl}(E)}$ such that $R(\varphi, E, \omega) \neq 0$. Then we have the following functional equation:

$$\widehat{L}(1-s,\pi) = (-1)^l N^{s-\frac{1}{2}} \widehat{L}(s,\pi).$$

- A transfer to $\mathrm{GL}_4(\mathbb{A})$.

(cf.) R. Schmidt, *Packet structure and paramodular forms.*, Trans. Amer. Math. Soc., **370** (2018), 3085–3112.

There exists an irreducible cuspidal automorphic representation Π of $\mathrm{GL}_4(\mathbb{A})$ such that

$$L(s,\pi) = L(s,\Pi).$$

Asymptotic formula(Future work)

Tsuzuki (2020) proved an asymptotic formula of a certain average of special values of spinor L-functions for full level Siegel cusp forms.

cf. M. Tsuzuki, *Weighted equidistribution theorem for Siegel modular forms of degree 2*, arXiv: 2103.07835.

Theorem (Tsuzuki (2020))

There exists a constant $C > 1$ depending only on D such that as $l \in 2\mathbb{Z}_{\geq 0}$ grows to $+\infty$,

$$\sum_{\pi \in \Pi_{\mathrm{cusp}}(l,1)} \Omega^{\pi}_{l,D,\omega} \cdot L(\tfrac{1}{2}, \pi) = 2P(l, D, \omega) + O(C^{-l})$$

where

$$P(l, D, \omega) := \begin{cases} L(1, \eta_D)\left(\frac{\Gamma'(l-1)}{\Gamma(l-1)} - \log(4\pi)^2\right) + L'(1, \eta_D) & (\omega = 1) \\ L(1, \mathcal{AI}(\omega)) & (\omega \neq 1) \end{cases}$$

$$\Omega^{\pi}_{l,D,\omega} := c_{l,D} \frac{|R(\varphi^0_\pi, E, \omega)|^2}{\langle \varphi^0_\pi, \varphi^0_\pi \rangle_{L^2}} \times \begin{cases} 1 & (\omega^2 = \mathbf{1}) \\ 2 & (\omega^2 \neq \mathbf{1}) \end{cases}$$

$$c_{l,D} := 4^{\frac{3}{2}-l} \pi^{\frac{7}{2}-2l} |D|^{\frac{3}{2}-l} \Gamma(l - \tfrac{3}{2}) \Gamma(l-2) w_D^{-1} h_D^{-1}.$$

We consider the average

$$\mathbb{I}^{(s)}(l,N) := \prod_{p|N}(p+1)^{-1}\cdot\left\{\sum_{\varphi\in\mathcal{B}(l,N)}\left\langle E^*(s,\omega), R(b_{\mathbb{R}}^{\theta})\varphi\right\rangle \overline{B^{T_\theta,\omega}(\varphi; b_{\mathbb{R}}^{\theta})}\right\}$$

($\mathcal{B}(l,N)$: an orthonormal basis of $S_l(\mathbf{K}_0(N))$)

By the basic identity (Piatetski-Shapiro (1997)), we have

$$\mathbb{I}^{(s)}(l,N) = \prod_{p|N}(p+1)^{-1}\sum_{\pi\in\Pi_{\text{cusp}}^{(T_\theta,\omega)}(l,N)}\left\{\sum_{\varphi\in\mathcal{B}_\pi(l,N)}\langle E^*(s,\omega), R(b_{\mathbb{R}}^{\theta})\varphi\rangle \overline{B^{T_\theta,\omega}(\varphi; b_{\mathbb{R}}^{\theta})}\right\}$$

($\mathcal{B}_\pi(l,N)$: an orthonormal basis of $S_l(\mathbf{K}_0(N))\cap\pi$)

We divide $\Pi_{\mathrm{cusp}}^{(T_\theta,\omega)}(l,N)$ into three parts:

$$\Pi_{\mathrm{cusp}}^{(T_\theta,\omega)}(l,N) = \Pi_{\mathrm{cusp}}^{(T_\theta,\omega)}(l,N)^{\mathrm{new,T}} \sqcup \Pi_{\mathrm{cusp}}^{(T_\theta,\omega)}(l,N)^{\mathrm{new,SK}} \sqcup \Pi_{\mathrm{cusp}}^{(T_\theta,\omega)}(l,N)^{\mathrm{old}}$$

where

- $\Pi_{\mathrm{cusp}}^{(T_\theta,\omega)}(l,N)^{\mathrm{old}}$: the subset consisting of all old forms.
- $\Pi_{\mathrm{cusp}}^{(T_\theta,\omega)}(l,N)^{\mathrm{new,SK}}$: the subset consisting of all newforms which is a Saito-Kurokawa lift from irred. cusp. rep'n of $\mathrm{PGL}_2(\mathbb{A})$.
- $\Pi_{\mathrm{cusp}}^{(T_\theta,\omega)}(l,N)^{\mathrm{new,T}} := \Pi_{\mathrm{cusp}}^{(T_\theta,\omega)}(l,N) - \Pi_{\mathrm{cusp}}^{(T_\theta,\omega)}(l,N)^{\mathrm{new,SK}} - \Pi_{\mathrm{cusp}}^{(T_\theta,\omega)}(l,N)^{\mathrm{old}}$.

We rewrite

$$\mathbb{I}^{(s)}(l,N) = \mathbb{I}^{(s)}(l,N)^{\mathrm{new,T}} + \mathbb{I}^{(s)}(l,N)^{\mathrm{new,SK}} + \mathbb{I}^{(s)}(l,N)^{\mathrm{old}}$$

Proposition1

$$\mathbb{I}^{(s)}(l,N)^{\mathrm{new},\mathrm{T}}$$
$$= \frac{2^{-2}|D|^{\frac{4-l}{2}} e^{-2\pi\sqrt{|D|}} N^{s+1}}{[\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(N)]} \sum_{\pi \in \Pi_{\mathrm{cusp}}^{(T_\theta,\omega)}(l,N)^{\mathrm{new},\mathrm{T}}} \frac{|R(\varphi_0^\pi, E, \omega)|^2}{\langle \varphi_0^\pi, \varphi_0^\pi \rangle_{L^2}} \widehat{L}(s+\tfrac{1}{2},\pi) \prod_{p|N} t_{\pi_p}$$

where

- $t_{\pi_p} = \begin{cases} \sigma_p & (\pi_p \text{ is of type VIb}) \\ p^{-1}\mathrm{tr}(T_0) & (\pi_p \text{ is of type IIIa}) \end{cases}.$

<u>Remark</u>
Boechere's conjecture (refined GGP conjecture (Liu, 2016)) states that

$$\frac{|R(\varphi_\pi^0, E, \omega)|^2}{\langle \varphi_\pi^0, \varphi_\pi^0 \rangle_{L^2}} = \frac{2^{4l-4}\pi^{2l+1}}{(2l-2)!} w_D^2 |D|^{l-1} \frac{L(\frac{1}{2}, \pi \times \mathcal{AI}(\omega^{-1}))}{L(1,\pi,\mathrm{Ad})}.$$

This formula has been proved by Furusawa and Morimoto (2022).

Proposition2

If $-\frac{1}{2} < \mathrm{Re}(s) < \frac{1}{2}$ and $N = q$ is a prime, then for any $\varepsilon > 0$,

$$\mathbb{I}^{(s)}(l,q)^{\mathrm{old}} = O(q^{\mathrm{Re}(s)-\frac{1}{2}}),$$
$$\mathbb{I}^{(s)}(l,q)^{\mathrm{new,SK}} = O(q^{-\frac{5}{4}-\frac{1}{2}\mathrm{Re}(s)+\varepsilon})$$

as $q \to \infty$.

<u>Outline</u>

$\mathbb{I}^{(s)}(l,N)^{\mathrm{old}} \longrightarrow$ Our results of local zeta integrals for old forms.

$\mathbb{I}^{(s)}(l,N)^{\mathrm{new,SK}} \longrightarrow$

- $\frac{|R(\varphi_\pi^0,E,\mathbf{1})|^2}{\langle\varphi_\pi^0,\varphi_\pi^0\rangle_{L^2}}\widehat{L}(s+\frac{1}{2},\pi) \sim \frac{L(\frac{1}{2},\pi_0\otimes\chi_D)L(s+\frac{1}{2},\pi_0)}{L(\frac{3}{2},\pi_0)L(1,\pi_0,\mathrm{Ad})}$
  if π is the Saito-Kurokawa lift of irred. cusp. rep'n π_0 of $\mathrm{PGL}_2(\mathbb{A})$.
  (Dickson, Pitale, Saha, Schmidt (2020))
- $L(s,\pi_0) \ll q^{\frac{1-\mathrm{Re}(s)}{2}+\varepsilon}, \quad (q \to \infty).$
- $L(1,\pi_0,\mathrm{Ad})^{-1} \ll \log(3+q), \quad (q \to \infty).$
  (Goldfeld, Hoffstein, Lieman(1994))