RECENT ADVANCES IN LANGLANDS' FUNCTORIALITY

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Hangzhou, August 2005

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Lecture 1. Langlands functoriality conjecture.

Let's start with a simple question: For which primes p does $x^2 - x - 1 \equiv 0 \pmod p$ have 2 solutions?

This is equivalent to the following question: Let $L = \mathbb{Q}(\sqrt{5})$. Then for which prime p splits completely in L?

The answer is given as follows: Let $P(L/\mathbb{Q}) = \{p|\, p \text{ splits completely in } L\}$. Then by the definition of Legendre symbol, $P(L/\mathbb{Q}) = \{p|\, \left(\frac{5}{p}\right) = 1\}$. By direct calculation, we can see that $P(L/\mathbb{Q})$ contains 11, 19, 29, 31, 41, 59, 61... If we look at the list carefully, we can see that they are primes congruent to 1, 4 modulo 5. Indeed, by quadratic reciprocity law, $\left(\frac{5}{p}\right) = \left(\frac{p}{5}\right)$. Hence $P(L/\mathbb{Q}) = \{p|\, p \equiv 1, 4 \pmod 5\}$.

More generally, let L/K be a Galois extension and let $P(L/K)$ be the set of prime ideals in K which split completely in L.

Fact. *$P(L/K)$ determines L completely.*

The goal of class field theory is to describe the Galois extension L in terms of data in K, namely, determine $P(L/K)$ in terms of data in K. When L/K is abelian, the answer is given completely by the class field theory. For example, $P(\mathbb{Q}(\zeta_m)/\mathbb{Q}) = \{p \equiv 1 \pmod m\}$, where ζ_m is a primitive mth roots of unity. But when L/K is not abelian, not much is known.

Example 1 (cf. [Se]). Consider $f(x) = x^3 - x - 1$. Here the discriminant is -23. Let L be the splitting field of f. Then $Gal(L/\mathbb{Q}) \simeq S_3$. If p is unramified, $f(x) \equiv 0 \pmod p$ has 0,1,3 solutions. Then $P(L/\mathbb{Q}) = \{p|\, f(x) \equiv 0 \pmod p \text{ has 3 solutions.}\}$. By computer calculation, we see that $P(L/\mathbb{Q}) = \{59, 101, 167, 173, ...\}$. On the other hand, $f(x) \equiv 0 \pmod p$ has 0 solutions when $p = 2, 3, 13, 29, 31, 41, ...$ It is hard to see the pattern. The pattern comes from modular forms. Let $\rho : S_3 \longrightarrow GL_2(\mathbb{C})$ be the 2-dimensional representation of S_3. Then we have the Artin L-function $L(s, \rho, L/\mathbb{Q})$. It is given by the Euler product

$$L(s, \rho, L/\mathbb{Q}) = \prod_p (1 - a_p p^{-s} + \left(\frac{p}{23}\right) p^{-2s})^{-1},$$

where $a_p = N_f(p) - 1$, and $N_f(p)$ is the number of solutions for $f(x) \equiv 0 \pmod p$. Here $L(s, \rho, L/\mathbb{Q}) = \frac{\zeta_E(s)}{\zeta(s)}$, where $E = \mathbb{Q}(\alpha)$, and α is a root of $f(x)$. This comes from the fact that $Ind_H^{S_3} 1 = 1 + \rho$. Here H is the Galois group of L/E, and $H \simeq \mathbb{Z}/2\mathbb{Z}$.

Langlands functoriality. *(1) $L(s, \rho, L/\mathbb{Q})$ is the L-function attached to a modular form of weight 1, level 23, with respect to the character $\epsilon(p) = \left(\frac{p}{23}\right)$. More* *precisely,* $L(s,\rho,L/\mathbb{Q}) = L(s,F)$,

$$F = q\prod_{k=1}^{\infty}(1-q^k)(1-q^{23k}) = \sum_{n=1}^{\infty} a_n q^n, \quad q = e^{2\pi i\tau}, \tau = x+iy$$

$$L(s,F) = \prod_p (1 - a_p p^{-s} + \left(\frac{p}{23}\right) p^{-2s})^{-1}.$$

(2) $P(L/\mathbb{Q}) = \{p|\, a_p = 2\}$.

Example 2 (cf. [Se]). $f(x) = x^4 - x - 1$. The Galois group is S_4. Let L be the splitting field of f and $E = \mathbb{Q}(\alpha)$, where α is a root of f. Let $H = Gal(L/E)$. Then $H \simeq S_3$, and $Ind_H^{S_4} 1 = 1 + \rho$, where $\rho : S_4 \longrightarrow GL_3(\mathbb{C})$ be a 3-dimensional representation. We have the Artin L-function $L(s,\rho,L/\mathbb{Q}) = \frac{\zeta_E(s)}{\zeta(s)}$.

In order to attach a modular form to ρ, we need a 2-dimensional representation. So consider an extension $\tilde{L}/L$ such that $Gal(\tilde{L}/\mathbb{Q}) = GL(2,\mathbb{F}_3)$. Here $(GL(2,\mathbb{F}_3) : S_4) = 2$. Then we have a 2-dimensional representation $\sigma : GL(2,\mathbb{F}_3) \longrightarrow GL_2(\mathbb{C})$ such that $\rho = Sym^2(\sigma)$. Langlands functoriality shows that σ is associated to F, a modular form of weight 1, and ρ is associated to $Sym^2(F)$, the symmetric square. Write $F = \sum_{n=1}^{\infty} a_n q^n$. Then $P(L/\mathbb{Q}) = \{p|\, a_p = \pm 2\}$.

Example 3. $f(x) = x^5 - x - 1$. The Galois group is S_5. It is not solvable. Let L be the splitting field of f and $E = \mathbb{Q}(\alpha)$, where α is a root of f. Let $H = Gal(L/E)$. Then $H \simeq S_4$, and $Ind_H^{S_5} 1 = 1 + \rho$, where $\rho : S_5 \longrightarrow GL_4(\mathbb{C})$ be a 4-dimensional representation. We have the Artin L-function $L(s,\rho,L/\mathbb{Q}) = \frac{\zeta_E(s)}{\zeta(s)}$. We do not have the answer for $P(L/\mathbb{Q})$. We only know its size. Chebotarev density theorem says that $P(L/\mathbb{Q})$ has Dirichlet density $\frac{1}{120}$.

Langlands functoriality conjecture. *There exists a cuspidal representation* $\pi = \otimes\pi_p$ *of* GL_4 *such that* $L(s,\rho,L/\mathbb{Q}) = L(s,\pi)$, *and* $P(L/\mathbb{Q}) = \{p|$ *Satake parameter of* π_p *is* $diag(1,1,1,1)\}$.

Weaker assertion (Artin conjecture). $L(s,\rho,L/\mathbb{Q})$ *is entire.*

We only know that $L(s,\rho,L/\mathbb{Q})$ has meromorphic continuation to all of $\mathbb{C}$ and satisfies a functional equation.

More generally, Langlands conjectured that given an irreducible representation $\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_n(\mathbb{C})$, there exists a cuspidal representation $\pi = \otimes\pi_p$ of GL_n such that $\rho(Frob_p)$ =Satake parameter of π_p. It is usually referred to as the strong Artin conjecture. A weaker assertion, known as Artin conjecture, claims that the Artin L-function $L(s,\rho)$ is entire.

Much effort has been made when $n = 2$. Let $\bar{\rho} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow PGL_2(\mathbb{C}) \simeq SO_3(\mathbb{C})$. Then $Im(\bar{\rho})$ is D_{2n} (dihedral), A_4 (tetrahedral), S_4 (octahedral), A_5 (icosahedral). The first three groups are solvable. It is a theorem of Langlands and Tunnell that for the case of A_4, S_4, the strong Artin conjecture is true, and it has been used by Andrew Wiles in his proof of Fermat's last theorem.

Cuspidal representations generalize classical modular forms (holomorphic and Maass forms). They can be understood as "direct summands" of the right regular representation of $G(\mathbb{A})$ on the Hilbert space $L^2(G(F)\backslash G(\mathbb{A}))$, where $\mathbb{A}$ is the ring of adeles. If π is a cuspidal representation, then we have a tensor product decomposition $\pi = \otimes \pi_v$, where v runs through all places of F; π_v is an irreducible, unitary representation of $G(F_v)$ for all v; π_v is spherical for almost all v, namely, it has the Satake parameter (semi-simple conjugacy class) $\{t_v\}$ in ${}^LG$ (the L-group of G).

When $G = GL_2$, $F = \mathbb{Q}$, there are two types of cuspidal representations: First, cuspidal representations attached to holomorphic cusp forms of weight k with respect to a congruence subgroup of $SL_2(\mathbb{Z})$; $\pi = \pi_f$, where $f(\tau) = \sum_{n=1}^{\infty} a_n n^{\frac{k-1}{2}} e^{2\pi i n\tau}, \tau = x + iy$. Then $\pi_f = \otimes \pi_p$, and the Satake parameter of π_p is $diag(\alpha_p, \beta_p)$, where $a_1 = 1, a_p = \alpha_p + \beta_p$.

Second, cuspidal representations attached to Maass cusp forms. They are eigenfunctions of the Laplacian; $y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f + (\frac{1}{4} - t^2)f = 0$, where $t \in i\mathbb{R}$ or $t \in \mathbb{R}$, $0 < t < \frac{1}{2}$. Then $f(\tau) = \sum_{n\neq 0} a_n |n|^{-\frac{1}{2}} W(n\tau)$, where $W(\tau) = y^{\frac{1}{2}} K_t(2\pi y) e^{2\pi i x}$ and K_t is the K-Bessel function.

Ramanujan conjecture. $|\alpha_p| = |\beta_p| = 1$.

Selberg conjecture. $t \in i\mathbb{R}$, *or* $\frac{1}{4} - t^2 \geq \frac{1}{4}$.

Theorem (Deligne, 1973). *Ramanujan conjecture is true for holomorphic cusp forms.*

Most general form of Langlands functoriality conjecture: Let H, G be two reductive groups. To each homomorphism of L-groups, $r : {}^LH \longrightarrow {}^LG$, there is associated a lift (transfer) of automorphic representations of H to automorphic representations of G which satisfy canonical properties.

Example 1. $H = \{e\}$, $G = GL_n$ over $\mathbb{Q}$. Then ${}^LH = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ and ${}^LG = GL_n(\mathbb{C})$. Then Langlands functoriality conjecture is the strong Artin conjecture.

Example 2. $H = GL_2, G = GL_{m+1}$. Let $Sym^m : GL_2(\mathbb{C}) \longrightarrow GL_{m+1}(\mathbb{C})$ be the mth symmetric power representation. We can prove the functoriality for $m \leq 4$. We will explain this in Lecture 3.

Example 3. $H = SO_{2n+1}, SO_{2n}, Sp_{2n}$, $G = GL_N$, where $N = 2n$ or $2n + 1$.

Then ${}^LH = Sp_{2n}(\mathbb{C}), SO_{2n}(\mathbb{C}), SO_{2n+1}(\mathbb{C})$, and $r : {}^LH \longrightarrow {}^LG$ is the embedding. This case is explained in detail by J. Cogdell.

Let us consider very particular Galois representations, namely, monomial representations: Given a 1-dimensional character χ of $H < G$, we can form an induced representation $Ind_H^G\chi$. Let $(G : H) = n$. Then $\rho = Ind_H^G\chi : G \longrightarrow GL_n(\mathbb{C})$. It is called monomial representation. We assume that ρ is irreducible. Let $G = Gal(L/\mathbb{Q})$, $H = Gal(L/E)$. Then by class field theory, χ can be considered as a grössencharacter of E of finite order, and $L(s, Ind_H^G\chi, L/\mathbb{Q}) = L(s,\chi)$, Hecke L-function. Hence $L(s, Ind_H^G\chi, L/\mathbb{Q})$ is entire, and Artin conjecture is true. However, the strong Artin conjecture is open. Namely, it is expected that there exists a cuspidal representation π of GL_n such that $L(s, Ind_H^G\chi, L/\mathbb{Q}) = L(s,\pi)$. We call π automorphic induction, we denote it by $I_E^{\mathbb{Q}}\chi$.

Fact. *Langlands functoriality for monomial representations implies Langlands functoriality for general Galois representations.*

Known cases: (1) $E/\mathbb{Q}$ is cyclic (solvable Galois). It is a special case of Arthur-Clozel. (2) $E/\mathbb{Q}$ is non-normal cubic extension. It is a result of Jacquet, Piatetski-Shapiro, and Shalika, a consequence of the converse theorem for GL_3. (3) The Galois closure of $E/\mathbb{Q}$ is solvable and χ is certain algebraic character. It is a result of M. Harris.

I will give two examples of non-normal $E/\mathbb{Q}$ whose Galois closure is not solvable; one is non-normal quintic extension, and the other, non-normal sextic extension.

Consider f, irreducible quintic polynomial with integer coefficients such that the Galois group is A_5. (J. Buhler's example: $f(x) = x^5 + 10x^3 - 10x^2 + 35x - 18$) Let L be the splitting field of f, $E = \mathbb{Q}(\alpha)$, where α is a root of f. Then $H = Gal(L/E) \simeq A_4$. Let K/E be such that $Gal(L/K)$ be the Klein 4-group and $Gal(K/E)$ is a cyclic group of order 3. Let χ be the grössencharacter of E attached to K/E by class field theory. It is considered as a character of H.

Theorem. *Under certain conditions, $Ind_H^{A_5}\chi$ is automorphic, namely, there exists a cuspidal representation $\pi = I_E^{\mathbb{Q}}\chi$ of GL_5 such that $L(s, Ind_H^{A_5}\chi, E/\mathbb{Q}) = L(s,\pi)$.*

Note that A_5 does not have a 2-dimensional representation. But there exists a central extension $L'/\mathbb{Q}$ such that $G' = Gal(L'/\mathbb{Q})$ has a 2-dimensional representation. (For J. Buhler's example, $(L' : A_5) = 20$ so that the central character has order 10.) Let $\rho : G' \longrightarrow GL_2(\mathbb{C})$. Such ρ factors through $\tilde{A}_5 \simeq SL(2,\mathbb{F}_5)$, where $Gal(\tilde{L}/\mathbb{Q}) = \tilde{A}_5$. In many cases, ρ is modular, namely, there exists a modular form f of weight 1 such that $L(s,\rho) = L(s,f)$. For example, R. Taylor showed that $f(x) = x^5 - 2x^3 + 2x^2 + 5x + 6$ gives rise to such ρ. The condition in the above theorem is that ρ is modular.

Here $Ind_H^{A_5}\chi$ is equivalent to $Sym^4(\rho)$, twisted by a character. Let ρ give rise to a cuspidal representation π of GL_2. Then $Sym^4\rho$ gives rise to $Sym^4\pi$. It is a cuspidal representation of GL_5 (the result of Kim, Kim-Shahidi. We will explain this in Lecture 3).

Second example (This is a new result which has not been published elsewhere.): Let N be the normalizer of a 5-Sylow subgroup in $\tilde{A}_5$. It is a group of order 20, isomorphic to $\mathbb{Z}/2\mathbb{Z} \times D_{10} \simeq D_{20}$. Let $\tilde{L}/\mathbb{Q}$ be such that $Gal(\tilde{L}/\mathbb{Q}) = \tilde{A}_5$, and $Gal(\tilde{L}/F) = N$. Then $F/\mathbb{Q}$ is a non-normal sextic extension. Take a quadratic character χ of N, non-trivial on its 2 component. Namely, let $\tilde{F}/F$ be a quadratic extension such that $Gal(\tilde{L}/\tilde{F}) \simeq D_{10}$. Then χ is the quadratic character attached to $\tilde{F}/F$ by class field theory. Then the induced character $Ind_N^{\tilde{A}_5}\chi$ is equivalent to $Sym^5(\rho)$, twisted by a character. (It is due to Serre.) If ρ gives rise to a cuspidal representation π of GL_2, $Sym^5\rho$ gives rise to $Sym^5\pi$. It is a cuspidal representation of GL_6 (the result of Kim-Shahidi, Wang). Hence we have

Theorem. *Suppose ρ is modular. Then $Ind_N^{\tilde{A}_5}\chi$ is automorphic, namely, there exists a cuspidal representation $\pi = I_F^{\mathbb{Q}}\chi$ of GL_6 such that $L(s, Ind_N^{\tilde{A}_5}\chi, F/\mathbb{Q}) = L(s,\pi)$.*

Another consequence is that $Sym^m(\pi)$ is automorphic for all m.

Irreducible representations of G' are equivalent to a twist of irreducible representations of $\tilde{A}_5$. We know all irreducible representations of $\tilde{A}_5$. Let σ, σ^τ be two 2-dimensional representations of $\tilde{A}_5$, where $\tau \in Aut(\mathbb{C})$ sends $\sqrt{5}$ to $-\sqrt{5}$. Then irreducible representations of $\tilde{A}_5$ are

(1) trivial (1 dim)
(2) σ, σ^τ (2 dim)
(3) $Sym^2(\sigma), Sym^2(\sigma^\tau)$ (3 dim)
(4) $Sym^3(\sigma) \simeq Sym^3(\sigma^\tau)$, $\sigma \otimes \sigma^\tau$ (4 dim)
(5) $Ind_H^{A_5}\chi \simeq Sym^4(\sigma) \simeq Sym^4(\sigma^\tau)$ (5 dim)
(6) $Sym^2(\sigma) \otimes \sigma^\tau \simeq \sigma \otimes Sym^2(\sigma^\tau) \simeq Sym^5(\sigma) \simeq Sym^5(\sigma^\tau)$ (6 dim)

Kim-Shahidi proved that $Sym^5(\pi)$ is an automorphic representation of GL_6. S. Wang proved that it is cuspidal. Now $Sym^m(\rho)$ is equivalent to a direct sum of irreducible representations of $\tilde{A}_5$ twisted by a character. Each direct summands is automorphic. Hence $Sym^m(\pi)$ is automorphic.

Lecture 2. Automorphic L-functions and Langlands-Shahidi method.

An L-function is a very special type of meromorphic functions of one complex variable. On the surface it is not clear why the L-functions play decisive roles. It was Riemann who introduced his zeta function in order to study the distribution of prime numbers: The prime number theorem says that $\pi(x) = \sum_{p \le x} 1 \sim \frac{x}{\ln x}$. It is equivalent to the fact that $\zeta(1+it) \neq 0$ for $t \in \mathbb{R}$. Dirichlet's theorem on arithmetic progression says that there are infinitely many primes in the arithmetic progression $an+b$, where $(a,b)=1$, $n=1,2,...$ It comes from the fact that $L(1,\chi) \neq 0$ for a Dirichlet character χ mod a.

An L-function is associated to the set A; arithmetic-geometric objects such as Galois groups, elliptic curves, and Shimura varieties. It is also associated to the set B; automorphic forms and representations. Langlands conjecture is that the set B contains the set A. The L-functions in the set B are called automorphic L-functions. Special case of such relationship for elliptic curves is called Taniyama-Shimura-Wiles theorem, i.e., elliptic curves over $\mathbb{Q}$ are modular.

Over $\mathbb{Q}$, an L-function which is associated to an object F takes the form of Euler product over all primes p, $L(s,F) = \prod_p L_p(s,F)$, $L_p(s,F) = \prod_{j=1}^m (1 - \alpha_j(p)p^{-s})^{-1}$ for almost all primes, where $\alpha_j(p) \in \mathbb{C}$. As a function of $s \in \mathbb{C}$, this product converges absolutely for $Re(s) >> 0$ and we can multiply out to get a series $L(s,F) = \sum a(n)n^{-s}$.

Example (1) Riemann zeta function $\zeta(s) = \prod_p (1-p^{-s})^{-1} = \sum_{n=1}^\infty n^{-s}$.

(2) Dirichlet L-function $L(s,\chi) = \prod_p (1-\chi(p)p^{-s})^{-1} = \sum_{n=1}^\infty \chi(n)n^{-s}$, where χ is a character of $(\mathbb{Z}/q\mathbb{Z})^\times$.

(3) $E: y^2 = x^3+ax+b, a,b \in \mathbb{Z}$. Let $N_E(p)$ be the number of solutions mod p. Let $a_E(p) = p - N_E(p)$. Then $L_p(s,E) = (1-a_E(p)p^{-s}+p^{1-2s})^{-1}$ for p non-singular. (Without normalization: In order to have a functional equation of the form $s \longmapsto 1-s$, we need to take $L_p(s,E) = (1-a_E(p)p^{-\frac{1}{2}-s}+p^{-2s})^{-1}$.)

More explicitly, let $E: y^2 = x^3 - 4x^2 + 16$. Then we have some numerical calculations (due to Silverman):

p	2	3	5	7	11	13	17	19	23	97	239	1327	3733	4817
$N_E(p)$	2	4	4	9	10	9	19	19	24	104	269	1259	3619	4949
$a_E(p)$	0	-1	1	-2	1	4	-2	0	-1	-7	-30	68	114	-132

Let $F = q\prod_{k=1}^\infty ((1-q^k)(1-q^{11k}))^2 = \sum_{n=1}^\infty b_n q^n$. Here F is a modular form of weight 2. Consider the L-function $L(s,F) = \sum_{n=1}^\infty b_n n^{-s}$. Then

$$L(s,F) = \prod_{p \neq 11} (1 - b_p p^{-s} + p^{1-2s})^{-1}(1-b_{11}11^{-s})^{-1}.$$

We have $a_E(p) = b_p$ for all $p \neq 2$.

(4) Automorphic L-functions. Let $\pi = \otimes \pi_v$ be a cuspidal representation of $G(\mathbb{A})$, where G is a split reductive group. Let ${}^LG$ be the L-group of G. Let $r : {}^LG \longrightarrow GL_N(\mathbb{C})$ be a finite dimensional representation of ${}^LG$. For $v \notin S$, π_v is spherical and it gives rise to a Satake parameter (semi-simple conjugacy class) $\{t_v\}$, $t_v \in {}^LG$. Form the local L-function $L(s, \pi_v, r) = det(I - r(t_v)q_v^{-s})^{-1}$. Let $L_S(s, \pi, r) = \prod_{v \notin S} L(s, \pi_v, r)$.

For example, let $\pi = \otimes \pi_v$ be a cuspidal representation of $GL_2(\mathbb{A})$. Let $diag(\alpha_v, \beta_v)$ be the Satake parameter of π_v. Let $Sym^m : GL_2(\mathbb{C}) \longrightarrow GL_{m+1}(\mathbb{C})$ be the mth symmetric power representation. Then

$$L(s, \pi_v, Sym^m) = \prod_{j=0}^{m} (1 - \alpha_v^{m-j} \beta_v^j q_v^{-s})^{-1}.$$

Central problems: (1) $L(s, F)$ has a meromorphic continuation to all of $\mathbb{C}$ and satisfies a function equation of the form: Let $\Lambda(s, F) = L(s, F) \times$ (some γ-factors and factors at bad places). Then $\Lambda(s, F) = \epsilon(s, F)\Lambda(1 - s, F')$, where F' is an object related to F such as a congredient representation. For example, $\Lambda(s) = \zeta(s)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) = \Lambda(1 - s)$.

(2) $\Lambda(s, F)$ is bounded in vertical strips.

(3) Grand Riemann Hypothesis; non-trivial zeros of $L(s, F)$ are all on $Re(s) = \frac{1}{2}$.

(4) Generalized Ramanujan conjecture: $|\alpha_j(p)| = 1$.

(5) Birch, Swinnerton-Dyer conjecture: Let $E/\mathbb{Q}$ be an elliptic curve. The order of vanishing of $L(s, E)$ at $s = 1$ (center of symmetry) is equal to the rank of the group of rational points on E.

Here (3) and (5) are two of seven one million dollar prize problems of Clay Math. Institute. There are other problems such as Siegel zeros (real zeros close to 1), and special value problems. For example, $L(1, \chi)$ contains a class number of a quadratic extension $K/\mathbb{Q}$, and χ is the non-trivial character of $Gal(K/\mathbb{Q})$.

Here even meromorphic continuation is not obvious. For example, it is clear that $\prod_{p \equiv 1 \pmod 4} (1 - p^{-s})^{-1}$ converges for $Re(s) > 1$. We can continue up to $Re(s) > 0$. But it is known that it has no meromorphic continuation to all of $\mathbb{C}$.

Let $\pi = \otimes \pi_v$ be a cuspidal representation of $GL_2(\mathbb{A})$. The L-function $L(s, \pi, Sym^m)$ was introduced by Langlands to solve Ramanujan and Sato-Tate conjecture. For example, if we know that $L(s, \pi, Sym^m)$ is absolutely convergent for $Re(s) > 1$ for all m, then $|\alpha_v^m| \leq q_v, |\beta_v^m| \leq q_v$ for all m. This implies that $|\alpha_v| \leq q_v^{\frac{1}{m}}, |\beta_v| \leq q_v^{\frac{1}{m}}$ for all m. Since $|\alpha_v \beta_v| = 1$, we have $|\alpha_v| = |\beta_v| = 1$. Kim-Shahidi proved meromorphic continuation and functional equation for $m \leq 9$.

Let $\pi = \otimes \pi_v$ be a cuspidal representation of $M(\mathbb{A})$, where π_v is spherical for $v \notin S$. Let $\psi = \otimes \psi_v$ be a fixed character of $\mathbb{A}/F$.

Conjecture (Langlands). *For $v \in S$, we can define a local factor $L(s, \pi_v, r)$ (of the form $P_v(q_v^{-s})^{-1}$, where P_v is a polynomial with constant 1), and a local ϵ-factor $\epsilon(s, \pi_v, r, \psi_v)$ (monomial in q_v^{-s}) such that $L(s, \pi, r) = \prod_{\text{all } v} L(s, \pi_v, r)$ has a meromorphic continuation to all of $\mathbb{C}$ and satisfies a functional equation $L(s, \pi, r) = \epsilon(s, \pi, r) L(1-s, \pi, \tilde{r})$, where $\tilde{r}(g) = {}^t r(g)^{-1}$.*

There are two ways of studying automorphic L-functions:

(1) method of Rankin-Selberg (integral representations); expresses L-functions as integrals of Eisenstein series, theta functions, etc. For example, Riemann proved that

$$\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) = \int_0^\infty x^{\frac{s}{2}} \frac{1}{2} (\theta(x) - 1) \, \frac{dx}{x}, \quad \theta(x) = \sum_{-\infty}^{\infty} e^{-n^2 \pi x}.$$

The Poisson summation formula gives rise to the functional equation of the theta function, $\theta(x^{-1}) = x^{\frac{1}{2}} \theta(x)$, and the functional equation of the Riemann zeta function follows.

(2) Langlands-Shahidi method; uses Eisenstein series attached to maximal parabolic subgroups.

We will briefly explain Langlands-Shahidi method. Let $P = MN$ be a maximal parabolic subgroup of G. Let π be a cuspidal representation of $M(\mathbb{A})$. Then we can form an induced representation, for $s \in \mathbb{C}$,

$$I(s, \pi) = Ind_P^G \pi \otimes exp(s\tilde{\alpha}, H_P(\,)),$$

where $\tilde{\alpha}$ is the fundamental weight corresponding to α, and α is a simple root such that P is associated to $\Delta - \{\alpha\}$. (Δ is the set of simple roots) For example, if $P = MN \subset Sp_{2n}$, $M \simeq GL_n$ (Siegel parabolic subgroup), then $I(s, \pi) = Ind_P^G \pi \otimes |det|^s$.

Given $f_s \in I(s, \pi)$, we define an Eisenstein series

$$E(s, f_s, g) = \sum_{\gamma \in P(F) \backslash G(F)} f_s(\gamma g).$$

Let $E_0(s, f_s, g) = \int_{N(F) \backslash N(\mathbb{A})} E(s, f_s, ng) \, dn$. It is called constant term. If P is self-conjugate (most cases),

$$E_0(s, f_s, g) = f_s(g) + M(s, \pi) f_s(g), \quad M(s, \pi) f_s(g) = \int_{N(\mathbb{A})} f_s(w_0^{-1} n g) \, dn,$$

where w_0 is a Weyl group element. $M(s,\pi)$ is called global intertwining operator from $I(s,\pi)$ to $I(-s,w_0(\pi))$.

Langlands proved that the poles of $E(s,f_s,g)$ and $M(s,\pi)$ are the same and they have meromorphic continuation to all of $\mathbb{C}$ and satisfy a functional equation $E(-s,M(s,\pi)f_s,g)=E(s,f_s,g)$.

Let $f_s=\otimes f_v$ and $I(s,\pi)=\otimes I(s,\pi_v)$. Then $M(s,\pi)=\otimes A(s,\pi_v,w_0)$, where

$$A(s,\pi_v,w_0)f_v(g)=\int_{N(F_v)} f_v(w_0^{-1}ng)\,dn.$$

It is called local intertwining operator. For almost all v, f_v is the unique K_v-fixed vector in $I(s,\pi_v)$ such that $f_v(e)=1$. Then Langlands proved (1971),

$$A(s,\pi_v,w_0)f_v=\prod_{i=1}^m \frac{L(is,\pi_v,r_i)}{L(1+is,\pi_v,r_i)}\tilde{f}_v,$$

where $\tilde{f}_v$ is the unique K_v-fixed vector in $I(-s,w_0(\pi_v))$ and r_i is certain irreducible finite-dimensional representation of ${}^LM$. Hence

$$M(s,\pi)=\prod_{i=1}^m \frac{L_S(is,\pi,r_i)}{L_S(1+is,\pi,r_i)}\otimes_{v\notin S}\tilde{f}_v\otimes\otimes_{v\in S}A(s,\pi_v,w_0)f_v.$$

By induction, this gives a meromorphic continuation of $L_S(s,\pi,r_i)$ for each i. But it does not give the desired functional equation.

At the suggestion of Langlands, Shahidi calculated ψ-Fourier coefficients of $E(s,f_s,g)$ for globally generic cuspidal representations, where ψ is a generic character of U. Here U is a maximal unipotent subgroup such that $B=TU$ is a Borel subgroup. Then $\psi_M=\psi|_{U_M}$ is a generic character of $U_M=U\cap M$. We say that $\pi=\otimes\pi_v$ is ψ_M-globally generic if $\int_{U_M(F)\backslash U_M(\mathbb{A})}\varphi(ug)\overline{\psi_M(u)}\,du\neq 0$ for a cuspidal function φ in the space of π. This implies that each π_v is locally generic, i.e., has a Whittaker model. If $\lambda_{\psi_v}(s,\pi_v)$ is the Whittaker functional for the space of $I(s,\pi_v)$, then by the uniqueness of Whittaker functional up to a constant,

$$\lambda_{\psi_v}(s,\pi_v)=C_{\psi_v}(s,\pi_v,w_0)\lambda_{\psi_v}(-s,w_0(\pi_v))A(s,\pi_v,w_0),$$

for some constant $C_{\psi_v}(s,\pi_v,w_0)\in\mathbb{C}$.

Consider ψ-Fourier coefficient of Eisenstein series

$$E_\psi(s,f_s,g)=\int_{U(F)\backslash U(\mathbb{A})} E(s,f_s,ug)\overline{\psi(u)}\,du.$$

Then Shahidi showed that

$$E_\psi(s, f_s, e) = \frac{\prod_{v\in S} W_v(e_v)}{\prod_{i=1}^m L_S(1+is, \pi, r_i)}, \quad W_v(e_v) = \lambda_{\psi_v}(s, \pi_v)(f_v).$$

Functional equation of Eisenstein series implies

$$\prod_{i=1}^m L_s(is, \pi, r_i) = \prod_{v\in S} C_{\psi_v}(s, \pi_v, w_0) \prod_{i=1}^m L_S(1-is, \pi, \tilde{r}_i).$$

Induction on i and detailed analysis of $C_{\psi_v}(s, \pi_v, w_0)$ lead to the definition of local factors $L(s, \pi_v, r_i), \epsilon(s, \pi_v, r_i, \psi_v)$ for $v \in S$, and the functional equation $L(s, \pi, r_i) = \epsilon(s, \pi, r_i) L(1-s, \pi, \tilde{r}_i)$ (Shahidi 1990).

But it had been thought that the location of poles is hard to obtain from Langlands-Shahidi method. Things changed by using spectral theory:

Langlands' Lemma. *Consider the residual spectrum* $L^2 = L^2_{dis}(G(F)\backslash G(\mathbb{A}))_{(M,\pi)}$.

(1) $L^2 = \{0\}$ *unless* $w_0\pi \simeq \pi$.
(2) L^2 *is spanned by the residues of* $E(s, f_s, g)$ *for* $s > 0$.
(3) *Suppose* $E(s, f_s, g)$ *has a pole at* $s = s_0$. *Then* $L^2 = \{\otimes_v J(s, \pi_v)\}$, *where* $J(s, \pi_v)$ *is the image of* $N(s, \pi_v, w_0)$, *the normalized local intertwining operator, namely,*

$$A(s, \pi_v, w_0) = \prod_{i=1}^m \frac{L(is, \pi_v, r_i)}{L(1+is, \pi_v, r_i)\epsilon(is, \pi_v, r_i)} N(s, \pi_v, w_0).$$

Then

$$M(s, \pi) = \prod_{i=1}^m \frac{L(is, \pi, r_i)}{L(1+is, \pi, r_i)\epsilon(is, \pi, r_i)} \otimes_v N(s, \pi_v, w_0). \tag{*}$$

(1) and (2) imply that $M(s, \pi)$ is holomorphic for $s > 0$ unless $w_0\pi \simeq \pi$.

Trick. $w_0(\pi \otimes \chi) \not\simeq \pi \otimes \chi$ *if* χ *is a grössencharacter which is highly ramified at one finite place.*

We denote $\pi \otimes \chi$ by π. Then $M(s, \pi)$ is holomorphic for $s > 0$.

We have to show that each local operator $N(s, \pi_v, w_0)$ is holomorphic and non-vanishing for $Re(s) \geq \frac{1}{2}$. This requires the study of representations of p-adic groups. The main ingredients are the following standard module conjecture and classification of discrete series representations.

Standard module conjecture. *Given a non-tempered generic representations π_v, there is a tempered data π_0 and a complex parameter Λ_0 which is in the corresponding positive Weyl chamber so that $\pi_v = I_{M_0}(\Lambda_0, \pi_0) = Ind_{M_0}^M(\pi_0 \otimes q_v^{<\Lambda_0, H_{P_0}^M(\ )>})$.*

According to Langlands, any non-tempered representation π_v can be written as a Langlands' quotient, namely, the quotient of $I_{M_0}(\Lambda_0, \pi_0)$. The above conjecture claims that if π_v is generic, π_v is $I_{M_0}(\Lambda_0, \pi_0)$ itself.

Theorem. *Except for certain cases (for the exact list, see [Ki2].) the local normalized intertwining operators $N(s, \pi_v, w_0)$ are holomorphic and non-vanishing for $Re(s) \geq \frac{1}{2}$.*

Let us illustrate it with an example of $GL_m \times SO_{2n+1}$. Let σ_v, π_v be generic unitary non-tempered representations of $GL_m(F_v), SO_{2n+1}(F_v)$, resp. By the standard module conjecture proved by Muić, σ_v, π_v can be written as follows:

$$\sigma_v = Ind\ (|det|^{\alpha_1}\sigma_1 \otimes \cdots \otimes |det|^{\alpha_p}\sigma_p \otimes \sigma_{p+1} \otimes |det|^{-\alpha_p}\sigma_p \otimes \cdots \otimes |det|^{-\alpha_1}\sigma_1),$$
$$\pi_v = Ind\ (|det|^{\beta_1}\tau_1 \otimes \cdots \otimes |det|^{\beta_q}\tau_q \otimes \tau_0),$$

where $0 < \alpha_p \leq \cdots \leq \alpha_1 < \frac{1}{2}$, $0 < \beta_q \leq \cdots \leq \beta_1$, and σ_i, τ_j, $i = 1, \cdots, p$, $j = 1, \cdots, q$, are discrete series representations of $GL(F_v)$, τ_0 (resp. σ_{p+1}) is a generic tempered representation of $SO_{2r+1}(F_v)$ (resp. $GL(F_v)$).

First we need to prove that $\beta_1 < 1$. Next we have

$$I(s, \sigma_v \otimes \pi_v) = Ind\ (|det|^{s+\alpha_1}\sigma_1 \otimes \cdots \otimes |det|^{s+\alpha_p}\sigma_p \otimes |det|^s\sigma_{p+1} \otimes$$
$$|det|^{s-\alpha_p}\sigma_p \otimes \cdots \otimes |det|^{s-\alpha_1}\sigma_1 \otimes |det|^{\beta_1}\tau_1 \otimes \cdots \otimes |det|^{\beta_q}\tau_q \otimes \tau_0).$$

Then $N(s, \sigma_v \otimes \pi_v, w_0)$ is a product of the rank-one operators

$$N(s \pm \alpha_i \pm \beta_j, \sigma_i \otimes \tau_j'), \quad N(2s \pm \alpha_i \pm \alpha_j, \sigma_i \otimes \sigma_j'), \quad N(s \pm \alpha_i, \sigma_i \otimes \tau_0),$$

where τ_j' is either τ_j or $\tilde{\tau}_j$ and σ_j' is either σ_j or $\tilde{\sigma}_j$. Here, for the sake of simplicity of notation, we have dropped the dependence of the local normalized operators on the Weyl group elements since these elements do not play any role in the argument.

For all s such that $Re(s) \geq \frac{1}{2}$, since $0 < \alpha_p \leq \cdots \leq \alpha_1 < \frac{1}{2}$ and $0 < \beta_q \leq \cdots \leq \beta_1 < 1$, we have $Re(s \pm \alpha_i \pm \beta_j) > -1$ and $Re(2s \pm \alpha_i \pm \alpha_j) > -1$. Hence the operators $N(s \pm \alpha_i \pm \beta_j, \sigma_i \otimes \tau_j')$ and $N(2s \pm \alpha_i \pm \alpha_j, \sigma_i \otimes \sigma_j')$, both corresponding to the case $GL_k \times GL_l \subset GL_{k+l}$, are holomorphic for $Re(s) \geq \frac{1}{2}$. For this result, one needs a classification of discrete series representations of GL_n. Next, we consider the operator $N(s \pm \alpha_i, \sigma_i \otimes \tau_0)$ which corresponds to the case

$GL_a \times SO_{2r+1} \subset SO_{2a+2r+1}$. Since $\sigma_i \otimes \tau_0$ is tempered, the operator $N(s, \sigma_i \otimes \tau_0)$ is holomorphic for $Re(s) \geq 0$. Consequently $N(s \pm \alpha_i, \sigma_i \otimes \tau_0)$ is holomorphic for $Re(s) \geq \frac{1}{2}$.

Thus we conclude $N(s, \sigma_v \otimes \pi_v, w_0)$ is holomorphic for $Re(s) \geq \frac{1}{2}$. Applying Zhang's lemma, we see that $N(s, \sigma_v \otimes \pi_v, w_0)$ is non-zero for $Re(s) \geq \frac{1}{2}$.

The result on the local intertwining operators together with $(*)$ implies that $\prod_{i=1}^m L(is, \pi, r_i)$ is holomorphic for $Re(s) \geq \frac{1}{2}$.

Next, from $E_\psi(s, f_s, e) = \frac{\prod_{v \in S} W_v(e_v)}{\prod_{i=1}^m L_S(1+is, \pi, r_i)}$, we see that $\prod_{i=1}^m L(1 + is, \pi, r_i)$ has no zeros for $Re(s) \geq 0$.

By using induction on i, we see that $L(s, \pi, r_i)$ is holomorphic for $Re(s) \geq \frac{1}{2}$ and has no zeros for $Re(s) \geq 1$. By the functional equation, we conclude that $L(s, \pi, r_i)$ is entire.

When $w_0(\pi) \simeq \pi$, we cannot conclude that $L(s, \pi, r_i)$ is entire. However, we can show

Theorem (Kim-Shahidi). *Suppose $m = 1$. Then $L_S(s, \pi, r_1)$ has at most a simple pole at $s = 1$. Suppose $m \geq 2$. Under the condition $L_S(2, \pi, r_2) \neq 0$, $L_S(s, \pi, r_1)$ has at most a simple pole at $s = 1$.*

Idea of Proof. It follows from the local-global principle: If $M(s, \pi)$ has a pole at $s = s_0$, then the quotient of $I(s_0, \pi_v)$ is unitary for all v. [If $M(s, \pi)$ has a pole at $s = s_0$, then the Eisenstein series has a pole at $s = s_0$, and its residues form a residual spectrum, that is, a direct summand of $L^2(G(F) \backslash G(\mathbb{A}))$. The local component of the residual spectrum is a quotient of $I(s, \pi_v)$, which is unitary.]

So if we can show that the quotient of $I(s, \pi_v)$ is not unitary at s_0, then $M(s, \pi)$ has no pole at s_0. Take π_v to be a spherical representation.

Lemma. *$I(s, \pi_v)$ is irreducible for $Re(s) \geq 2$.*

Proof. J.S. Li showed that $I(s, \pi_v)$ is irreducible at $s = s_0$ if and only if $\prod_{i=1}^m L(1 - is, \pi_v, r_i)$ has no poles at $s = s_0$. Here $L(s, \pi_v, r_i) = \prod_j (1 - \alpha_j q_v^{-s})^{-1}$. Shahidi showed that $|\alpha_j|, |\alpha_j|^{-1} < q_v$. So if $Re(s) \geq 2$, $L(s, \pi_v, r_i)$ has no poles.

Lemma. *$I(s, \pi_v)$ is not unitary for $Re(s) \geq 2$.*

Proof. $I(s, \pi_v)$ is not unitary for $Re(s) >> 0$. Unitarity is preserved up to a point of reducibility. So $I(s, \pi_v)$ is not unitary for $Re(s) \geq 2$.

We have proved that $M(s, \pi)$ is holomorphic for $Re(s) \geq 2$. Since $A(s, \pi_v, w_0)$ is non-vanishing, $\prod_{i=1}^m \frac{L_S(is, \pi, r_i)}{L_S(1+is, \pi, r_i)}$ is holomorphic for $Re(s) \geq 2$. Shahidi showed that $L_S(s,\pi,r_i)$ is absolutely convergent for $Re(s)>2$. So $L_S(s,\pi,r_1)$ is holomorphic for $Re(s)\geq 2$. Since $M(s,\pi)$ has at most a simple pole at $s=1$, $\prod_{i=1}^m \frac{L_S(is,\pi,r_i)}{L_S(1+is,\pi,r_i)}$ has at most a simple pole at $s=1$. So $L_S(s,\pi,r_1)L_S(2s,\pi,r_2)$ has at most a simple pole at $s=1$. So under the assumption $L_S(2,\pi,r_2)\neq 0$, $L_S(s,\pi,r_1)$ has at most a simple pole at $s=1$.

Corollary. *Let π be a cuspidal representation of $GL_6(\mathbb{A}_F)$. Then the exterior cube L-function $L(s,\pi,\wedge^3)$ (degree 20 L-function) has at most a simple pole at $s=1$.*

Proof. Apply the theory to $\mathbf{M}\subset E_6$, where the derived group of $\mathbf{M}$ is SL_6.

We conclude with a discussion of some observation: Suppose $m\geq 2$.

Conjecture. $\prod_{i=1}^m L_S(1+is,\pi,r_i)$ *has at most a simple pole at $s=0$.*

Since $L(s,\pi,r_i)$, $i\geq 3$, is holomorphic for $Re(s)>0$, this means that only one of $L(s,\pi,r_1), L(s,\pi,r_2)$ can have a pole at $s=1$.

Examples: (1) Consider $GL_m\times SO_{2n+1}\subset SO_{2(m+n)+1}$. The L-functions in the constant term of the Eisenstein series are

$$L(s,\sigma\times\pi)L(2s,\sigma,Sym^2).$$

If $L(s,\sigma\times\pi)$ has a pole at $s=1$, then $\sigma\simeq\tilde{\sigma}$ and $L(s,\sigma,Sym^2)$ is holomorphic at $s=1$ by the conjecture. Since $L(s,\sigma\times\sigma)=L(s,\sigma,Sym^2)L(s,\sigma,\wedge^2)$, $L(s,\sigma,\wedge^2)$ has a pole at $s=1$. This has been proved by Ginzburg-Rallis-Soudry.

(2) Consider $\mathbf{M}=GL_2\subset G_2$ (the maximal Levi subgroup attached to the long simple root in the exceptional group of type G_2). The L-functions in the constant term of the Eisenstein series are

$$L(s,\pi,Sym^3\otimes\omega_\pi^{-1})L(2s,\omega_\pi).$$

If $L(s,\pi,Sym^3\otimes\omega_\pi^{-1})$ has a pole at $s=1$, π is monomial and hence $\omega_\pi\neq 1$. If $\omega_\pi=1$, then $L(s,\pi,Sym^3\otimes\omega_\pi^{-1})$ is holomorphic at $s=1$ (the result of Ikeda).

Some questions in the Langlands-Shahidi method:

(1) Prove the above conjecture.

(2) Unitarity criterion for $I(s,\pi_v)$ when π_v is a spherical representation. This will give the location of poles of $L(s,\pi,r_i)$.

(3) Find the criterion for the pole of $L(s,\pi,r_1)$ at $s=1$. Goldberg-Shahidi have been studying the criterion for the pole of the local L-function at $s=0$ in

terms of twisted endoscopy. We want a global analogue. For example, we want to find a criterion of the pole in terms of some period integrals. For example, Flicker showed that $L(s, \sigma, r_A)$ has a pole at $s = 1$ precisely when σ is $GL_k(\mathbb{A}_F)$-distinguished, namely, the central character ω_σ satisfies the condition $\omega_\sigma|_{\mathbb{A}_F^\times} = 1$ and there exists a cuspidal function φ such that

$$\int_{GL_k(F)Z(\mathbb{A}_F)\backslash GL_k(\mathbb{A}_F)} \varphi(h)\, dh \neq 0,$$

where Z is the center of GL_k.

(4) Develop the theory for Kac-Moody groups (infinite-dimensional groups).

Lecture 3. Functoriality of symmetric powers of cuspidal representations of GL_2.

Let $\pi = \otimes \pi_v$ be a cuspidal representation of GL_2 over a number field F. Let $Sym^m : GL_2(\mathbb{C}) \longrightarrow GL_{m+1}(\mathbb{C})$ be the mth symmetric power representation. The local Langlands correspondence, proved by Langlands for archimedean places, and by Harris-Taylor, Henniart for p-adic places, says that π_v is parametrized by $\phi_v : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow GL_2(\mathbb{C})$. Consider the composition $Sym^m(\phi_v) : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow GL_{m+1}(\mathbb{C})$. By the local Langlands correspondence, $Sym^m(\phi_v)$ is associated to an irreducible, admissible representation $Sym^m(\pi_v)$ of $GL_{m+1}(F_v)$. Let $Sym^m(\pi) = \otimes Sym^m(\pi_v)$. It is an irreducible, admissible representation of $GL_{m+1}(\mathbb{A})$.

Langlands functoriality conjecture. *$Sym^m(\pi)$ is automorphic.*

Theorem. *(1) (Gelbart-Jacquet) $Sym^2(\pi)$ is an automorphic representation of GL_3. It is not cuspidal if and only if π is of dihedral type, i.e., $\pi \simeq \pi \otimes \chi$ for $\chi \neq 1$.*

(2) (Kim-Shahidi) $Sym^3(\pi)$ is an automorphic representation of GL_4. It is not cuspidal if and only if π is either of dihedral type, or of tetrahedral type, i.e., $Sym^2(\pi)$ is cuspidal and $Sym^2(\pi) \simeq Sym^2(\pi) \otimes \chi$ for $\chi \neq 1$.

(3) (Kim) $Sym^4(\pi)$ is an automorphic representation of GL_5.

(4) (Kim-Shahidi) $Sym^4(\pi)$ is not cuspidal if and only if π is either of dihedral type, or of tetrahedral type, or of octahedral type, i.e., $Sym^3(\pi)$ is cuspidal and $Sym^3(\pi) \simeq Sym^3(\pi) \otimes \chi$ for $\chi \neq 1$.

3.1 Proof of functoriality of $Sym^2(\pi)$.

In order to illustrate our method, let us prove the functoriality of $Sym^2(\pi)$ using our method. Let $\phi = Ad : GL_2(\mathbb{C}) \longrightarrow SO_3(\mathbb{C}) \subset GL_3(\mathbb{C})$ be the adjoint representation. Let $\pi = \otimes_v \pi_v$ be a cuspidal representation of $GL_2(\mathbb{C})$. By the local Langlands' correspondence, we have $Ad(\pi) = \otimes_v Ad(\pi_v)$, irreducible admissible representation of $GL_3(\mathbb{A})$. In this case, note that, for a grössencharacter χ,

$$L(s, \pi \times \tilde{\pi}) = L(s, Ad(\pi))L(s,1), \quad L(s, (\pi \otimes \chi) \times \tilde{\pi}) = L(s, Ad(\pi) \otimes \chi)L(s,\chi).$$

Hence $L(s, Ad(\pi_v) \otimes \chi_v)$ and $\gamma(s, Ad(\pi_v) \otimes \chi_v, \psi_v)$ are Artin L and γ-factors. Here $L(s, Ad(\pi) \otimes \chi)$ appears in the Langlands-Shahidi method for $P = MN \subset Sp(4)$, where $M \simeq GL_1 \times SL_2$: Take π_0 to be any irreducible constituent of $\pi|_{SL_2(\mathbb{A})}$ and consider $\Sigma = \chi \otimes \pi_0$. Then (M, Σ) gives rise to the fact that $L(s, Ad(\pi) \otimes \chi)$ is entire if $\chi^2 \neq 1$. We recall

Converse theorem. *Suppose $\Pi = \otimes \Pi_v$ is an irreducible, admissible representation of GL_N such that $\omega_\Pi = \otimes \omega_{\Pi_v}$ is a grössencharacter. Let S be a finite set* *of finite places and let* $\mathcal{T}^S(m)$ *be the set of all cuspidal representations of* GL_m *that are unramified at* S*. Suppose* $L(s, \sigma \times \Pi)$ *is nice (entire, functional equation, bounded in vertical strips) for all* $\sigma \in \mathcal{T}^S(m) \otimes \chi$, $m = 1, ..., N-2$, *where* χ *is a grössencharacter which is highly ramified at* S*. Then there exists an automorphic representation* Π' *of* GL_N *such that* $\Pi'_v \simeq \Pi_v$ *for* $v \notin S$*.*

Apply the converse theorem twice to $Ad(\pi) = \otimes_v Ad(\pi_v)$, and $S_1 = \{v_1\}$, $S_2 = \{v_2\}$, where v_1, v_2 are any finite places. Then we obtain Π_1, Π_2 automorphic representations of $GL_3(\mathbb{A})$ such that $\Pi_{1v} \simeq Ad(\pi_v)$ for all $v \neq v_1$, and $\Pi_{2v} \simeq Ad(\pi_v)$ for all $v \neq v_2$. So by the strong multiplicity one, $\Pi_1 \simeq \Pi_2 \simeq Ad(\pi)$. Therefore $Ad(\pi)$ is an automorphic representation. By the classification of automorphic representations of GL_n, $Ad(\pi)$ is equivalent to a subquotient of

$$Ind\, \sigma_1 |det|^{r_1} \otimes \cdots \otimes \sigma_k |det|^{r_k},$$

where σ_i's are (unitary) cuspidal representations of GL_{n_i} and $r_i \in \mathbb{R}$.

We use the weak Ramanujan property: Let $\pi = \otimes \pi_v$ be a cuspidal representation of GL_n, and π_v is spherical with the Satake parameter $diag(\alpha_{1v}, ..., \alpha_{nv})$. We say π satisfies the weak Ramanujan property if given $\epsilon > 0$, there exists T_ϵ, a density zero set such that $\max_i\{|\alpha_{iv}|, |\alpha_{iv}|^{-1}\} \leq q_v^\epsilon$ for $v \notin T_\epsilon$.

Lemma. *Cuspidal representations of* GL_2, GL_3 *satisfy the weak Ramanujan property.*

Since π satisfies the weak Ramanujan, so does $Ad(\pi)$. Hence $r_1 = \cdots = r_k = 0$. From the relation $L(s, (\pi \otimes \chi) \times \tilde{\pi}) = L(s, Ad(\pi) \otimes \chi) L(s, \chi)$, the left hand side has a pole at $s = 1$ if and only if $\pi \otimes \chi \simeq \pi$, i.e., π is monomial. Then the right hand side has a pole. So $Ad(\pi) \simeq \chi \boxplus \pi'$, where π' is an automorphic representation of GL_2. Hence $Ad(\pi)$ is not cuspidal if and only if π is of dihedral type.

3.2 Proof of functoriality of $Sym^3(\pi)$*.*

This is obtained indirectly from the functorial product associated with the tensor product map

$$GL_2(\mathbb{C}) \times GL_3(\mathbb{C}) \longrightarrow GL_6(\mathbb{C}).$$

Let π_1, π_2 be cuspidal representations of GL_2, GL_3, resp. For each place v, π_{iv}, $i = 1, 2$, are parametrized by $\phi_{iv} : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow GL_{i+1}(\mathbb{C})$, $i = 1, 2$. Then $\phi_{1v} \otimes \phi_{2v} : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow GL_6(\mathbb{C})$. By the local Langlands correspondence, $\phi_{1v} \otimes \phi_{2v}$ gives rise to an irreducible, admissible representation $\pi_{1v} \boxtimes \pi_{2v}$ of $GL_6(F_v)$. Let $\pi_1 \boxtimes \pi_2 = \otimes(\pi_{1v} \boxtimes \pi_{2v})$. It is an irreducible, admissible representation of $GL_6(\mathbb{A})$.

Theorem (Kim-Shahidi). $\pi_1 \boxtimes \pi_2$ *is automorphic.*

Let π be a cuspidal representation of GL_2, and let $Ad(\pi)$ be the adjoint square, i.e., $Ad(\pi) = Sym^2(\pi) \otimes \omega_\pi^{-1}$, where ω_π is the central character. Then $\pi \boxtimes Ad(\pi) = (Sym^3(\pi) \otimes \omega_\pi^{-1}) \boxplus \pi$. Here $\boxplus$ is the isobaric sum, and it denotes the unitary induction $Ind_{GL_4 \times GL_2}^{GL_6} (Sym^3(\pi) \otimes \omega_\pi^{-1}) \otimes \pi$.

Proof of the theorem. We apply the converse theorem to $\pi_1 \boxtimes \pi_2$. Let S be a finite set of finite places such that π_{1v}, π_{2v} are spherical for $v \notin S$, $v < \infty$. We need the triple product L-functions

$$L(s, \sigma \times (\pi_1 \boxtimes \pi_2)) = L(s, \sigma \times \pi_1 \times \pi_2),$$

where σ is a cuspidal representation of GL_m, $m = 1, 2, 3, 4$. These are available from Langlands-Shahidi method:

(1) $m = 1$: Rankin-Selberg L-function of $GL_2 \times GL_3$
(2) $m = 2$: $D_5 - 2$ case. Use $Spin(10)$.
(3) $m = 3$: $E_6 - 1$ case. Use simply connected E_6.
(4) $m = 4$: $E_7 - 1$ case. Use simply connected E_7.

Functional equation: due to Shahidi (1990)

Bounded in vertical strips: due to Gelbart and Shahidi (2001)

Entire: trick to use χ which is highly ramified at one finite place

By applying the converse theorem, we obtain an automorphic representation $\Pi = \otimes \Pi_v$ of GL_6 such that $\Pi_v \simeq \pi_{1v} \boxtimes \pi_{2v}$ for $v \notin S$. By classification of automorphic representations of GL_N (due to Jacquet-Shalika), Π is equivalent to a subquotient of $Ind |det|^{r_1} \sigma_1 \otimes \cdots \otimes |det|^{r_k} \sigma_k$, where $r_1, ..., r_k \in \mathbb{R}$ and σ_i's are cuspidal representations of GL_{n_i}.

We need: $r_1 = \cdots = r_k = 0$.

We use the weak Ramanujan property. Let $diag(\alpha_1, \beta_1), diag(\alpha_2, \beta_2, \gamma_2)$ be Satake parameters for π_{1v}, π_{2v}, resp. Then the Satake parameter of Π_v is $diag(\alpha_1\alpha_2, \alpha_1\beta_2, \alpha_1\gamma_2, \beta_1\alpha_2, \beta_1\beta_2, \beta_1\gamma_2)$. This implies that Π satisfies the weak Ramanujan property. Hence $r_1 = \cdots = r_k = 0$.

Proving that $\Pi_v \simeq \pi_{1v} \boxtimes \pi_{2v}$ for $v \in S$ requires extra work using base change. The most difficult case is when $v|2$, π_{1v} is an extraordinary supercuspidal representation of $GL_2(F_v)$, and π_{2v} is a supercuspidal representation of $GL_3(F_v)$ attached to a non-normal cubic extension K/F_v, namely, π_{2v} is associated to $Ind_{W_K}^{W_{F_v}} \chi$, where χ is a character of W_K. This has been done in the appendix by Bushnell and Henniart.

3.3 Proof of functoriality of $Sym^4(\pi)$.

This is obtained indirectly from the functoriality of the exterior square of GL_4: Let π be a cuspidal representation of GL_4, and let $\wedge^2 : GL_4(\mathbb{C}) \longrightarrow GL_6(\mathbb{C})$ be the map given by the exterior square. We can define, by the local Langlands correspondence, the irreducible admissible representation $\wedge^2\pi = \otimes \wedge^2 \pi_v$ of $GL_6(\mathbb{A})$.

Theorem (Kim). *Let T be a set of places such that $v|2,3$ and π_v is a supercuspidal representation of $GL_4(F_v)$. Then there exists an automorphic representation $\Pi = \otimes\Pi_v$ of GL_6 such that $\Pi_v \simeq \wedge^2\pi_v$ for $v \notin T$.*

Let π be a cuspidal representation of GL_2. Then we can prove

$$\wedge^2(Sym^3(\pi) \otimes \omega_\pi^{-1}) = (Sym^4(\pi) \otimes \omega_\pi^{-1}) \boxplus \omega_\pi.$$

Proof of the theorem. We apply the converse theorem to $\wedge^2\pi$. We need the L-functions

$$L(s, \sigma \times \wedge^2\pi) = L(s, \sigma \otimes \pi, \rho_m \otimes \wedge^2\rho_4),$$

where $\rho_m : GL_m(\mathbb{C}) \longrightarrow GL_m(\mathbb{C})$ is the standard representation and σ is a cuspidal representation of GL_m, $m = 1, 2, 3, 4$.

These L-functions show up in $D_n - 3$ case for $n = 4, 5, 6, 7$. The reason there is an exceptional set T is because we could not prove the identity

$$\begin{aligned} &L(s, \eta_v \otimes \wedge^2\pi_v) = L(s, \pi_v, \wedge^2 \otimes \eta_v), \\ (**) \qquad &\gamma(s, \eta_v \otimes \wedge^2\pi_v, \psi_v) = \gamma(s, \pi_v, \wedge^2 \otimes \eta_v, \psi_v), \end{aligned}$$

for $v \in T$. The left hand sides are defined by Rankin-Selberg method. The right hand sides are defined by Langlands-Shahidi method as normalizing factors of intertwining operators.

It is conjectured that when η_v is highly ramified, we have the following

Stability of γ-factors. *Let π_{1v}, π_{2v} be two irreducible admissible representations of $GL_4(F_v)$ with the same central character. Then for every highly ramified character η_v,*

$$\begin{aligned} L(s, \pi_{1v}, \wedge^2 \otimes \eta_v) &= L(s, \pi_{2v}, \wedge^2 \otimes \eta_v) \equiv 1, \\ \gamma(s, \pi_{1v}, \wedge^2 \otimes \eta_v, \psi_v) &= \gamma(s, \pi_{2v}, \wedge^2 \otimes \eta_v, \psi_v). \end{aligned}$$

Since it is not available, we have to use the descent method of Ramakrishnan. Namely, we first look at a good case when T is empty. When T is not empty, we use the observation of Henniart that a supercuspidal representation of $GL_n(F_v)$ becomes a principal series after a solvable base change.

Theorem (Ramakrishnan). *Fix $n, p \in \mathbb{N}$ with p prime. Let F a number field, $\{K_j | j \in \mathbb{N}\}$ a family of cyclic extensions of F with $[K_j : F] = p$, and for each $j \in \mathbb{N}$, π_j a cuspidal automorphic representation of $GL_n(\mathbb{A}_{K_j})$. Suppose that, given $j \in \mathbb{N}$,*

$$(\pi_j)_{K_j K_r} \simeq (\pi_r)_{K_j K_r},$$

for almost all $r \in \mathbb{N}$. Then there exists a unique cuspidal automorphic representation π of $GL_n(\mathbb{A}_F)$ such that

$$(\pi)_{K_j} \simeq \pi_j,$$

for all but a finite number of j.

For the functoriality of the symmetric fourth, we can prove $(**)$ directly and hence we do not need the descent method of Ramakrishnan.

Lecture 4. Applications.

4.1 Properties of symmetric power L-functions.

For simplicity, let π be a cuspidal representation of GL_2 with the trivial central character and suppose that $Sym^4(\pi)$ is cuspidal. Then we have

(1) $L(s,\pi,Sym^m), m\le 9$, has a meromorphic continuation to all of $\mathbb{C}$ and satisfies a functional equation.
(2) $L(s,\pi,Sym^m), m\le 8$, is invertible for $Re(s)\ge 1$ (no poles and no zeros).
(3) $L(s,\pi_v,Sym^9)$ is holomorphic for $Re(s)\ge 1$ for any v. If π_v is spherical with $diag(\alpha_v,\beta_v)$ as the Satake parameter, then $q_v^{-\frac{1}{9}}<|\alpha_v|,|\beta_v|<q_v^{\frac{1}{9}}$.
(4) $L(s,\pi,Sym^3), L(s,\pi,Sym^4)$ are entire.

It is conjectured that the multiplicity of poles of $L(s,\pi,Sym^m)$ at $s=1$ determines π. For example, if π is a cuspidal representation of icosahedral type, $L(s,\pi,Sym^m)$ has a simple pole at $s=1$ when $m=12$.

Let us show how the properties of $L(s,\pi,Sym^9)$ are proved in order to indicate the limitation of our method: Consider the case E_8-2. Let M be a maximal Levi subgroup in the exceptional group of type E_8 whose derived group M_D is $SL_4\times SL_5$. Let π_i, $i=1,2$, be cuspidal representations of $GL_4(\mathbb{A})$ and $GL_5(\mathbb{A})$ with central characters ω_i, $i=1,2$, resp. Let $\pi_{i0}, i=1,2$, be irreducible constituents of $\pi_1|_{SL_4(\mathbb{A})}$ and $\pi_2|_{SL_5(\mathbb{A})}$, resp. Then $\Sigma=\omega_1^5\omega_2^8\otimes\pi_{10}\otimes\pi_{20}$ can be considered as a cuspidal representation of $M(\mathbb{A})$. Applying Langlands-Shahidi method, we then get the L-function $L(s,\pi_1\otimes\pi_2,\rho_4\otimes\wedge^2\rho_5)$ as our first L–function. In fact, there are five L-functions in the constant term of the Eisenstein series; namely

$$L_S(s,\Sigma,r_1)=L_S(s,\pi_1\otimes\pi_2,\rho_4\otimes\wedge^2\rho_5);$$
$$L_S(s,\Sigma,r_2)=L_S(s,\pi_1\otimes(\tilde{\pi}_2\otimes\omega_2),\wedge^2\rho_4\otimes\rho_5);$$
$$L_S(s,\Sigma,r_3)=L_S(s,\tilde{\pi}_1\times(\pi_2\otimes\omega_1\omega_2));$$
$$L_S(s,\Sigma,r_4)=L_S(s,\tilde{\pi}_2,\wedge^2\rho_5\otimes\omega_1\omega_2^2);$$
$$L_S(s,\Sigma,r_5)=L_S(s,\pi_1\otimes\omega_1\omega_2^2).$$

Each of the L-functions, especially, $L_S(s,\Sigma,r_1)$ has a meromorphic continuation and satisfies a standard functional equation.

We apply the above to $\pi_1=A^3(\pi)$ and $\pi_2=Sym^4(\pi)$. By standard calculations, we have

$$L_S(s,\pi_1\otimes\pi_2,\rho_4\otimes\wedge^2\rho_5)=L_S(s,\pi,Sym^9)L_S(s,\pi,Sym^7\otimes\omega_\pi)L_S(s,\pi,Sym^5\otimes\omega_\pi^2)^2$$
$$L_S(s,Sym^3(\pi)\otimes\omega_\pi^3)^2L_S(s,\pi\otimes\omega_\pi^4).$$

Meromorphic continuation and functional equation of $L_S(s,\pi,Sym^9)$ now follow from those of $L_S(s,\pi_1\otimes\pi_2,\rho_4\otimes\wedge^2\rho_5)$.

The group of type E_8 is the largest finite dimensional exceptional group. The groups of type E_n, $n \geq 9$, are not finite dimensional, and they are called Kac-Moody groups.

4.2 Refined Ramanujan and Selberg bounds.

Let $\pi = \otimes \pi_p$ be a cuspidal automorphic representation for $GL_2(\mathbb{A}_\mathbb{Q})$. For p a prime at which π_p is spherical, let $diag(\alpha_p, \beta_p)$ be the corresponding Satake parameter. Let $Sym^4(\pi)$ be the symmetric fourth, the cuspidal representation of GL_5. (We assume that it is cuspidal since if not, π is associated to a Galois representation, and it satisfies the Ramanujan conjecture.) Then from the fact that $L(s, Sym^4(\pi), Sym^2)$ is absolutely convergent for $Re(s) > 1$, we can show that

Theorem (Kim-Sarnak). $p^{-\frac{7}{64}} \leq |\alpha_p| \leq p^{\frac{7}{64}}$. *Let* $\lambda_1(\Gamma)$ *be the smallest (nonzero) eigenvalue of the Laplacian on* $\Gamma\backslash\mathbb{H}$*, where* Γ *is a congruence subgroup of* $SL_2(\mathbb{Z})$*. Then*

$$\lambda_1(\Gamma) \geq \frac{975}{4096} \approx 0.238...$$

4.3 Density of tempered places.

Let π be a cuspidal representation of GL_2. Let $S(\pi)$ be the set of places where π_v is tempered. Ramanujan conjecture predicts that $S(\pi)$ is the set of all places. Ramakrishnan proved, using the functoriality of symmetric square, that $S(\pi)$ has lower Dirichlet density$\geq \frac{9}{10}$. By using the functoriality of symmetric cube and symmetric fourth, we can show

$$\underline{\delta}(S(\pi)) \geq \frac{34}{35}.$$

Here $\underline{\delta}(S)$ is the lower Dirichlet density, defined by

$$\underline{\delta}(S) = \underline{\lim}_{s\to 1+} \frac{\sum_{v\in S} q_v^{-s}}{-\log(s-1)}.$$

4.4 Special case of Sato-Tate conjecture.

Let $\pi = \otimes \pi_v$ be a cuspidal representation of GL_2 with the trivial central character. We also assume the Ramanujan conjecture. Let $diag(\alpha_v, \beta_v)$ be the Satake parameter for the spherical π_v. Let $a_v(\pi) = \alpha_v + \beta_v$. Since $|a_v(\pi)| \leq 2$, write $a_v(\pi) = 2\cos\theta_v$, $0 \leq \theta_v < \pi$. Sato and Tate conjectured that θ_v is distributed in the following way:

$$\#\{v : q_v \leq x, \theta_v \in (a,b)\} \sim \left(\frac{2}{\pi}\int_a^b \sin^2\theta \, d\theta\right) \pi_F(x),$$

as $x \to \infty$. Here $\pi_F(x) = \#\{v : q_v \leq x\} \sim \frac{x}{\ln x}$. The weaker version of Sato-Tate conjecture claims that the set $\#\{v : \theta_v \in (a,b)\}$ has positive lower density for any $a < b$.

Theorem. *For every $\epsilon > 0$, there exist T^+, T^- of positive lower density such that $a_v(\pi) > 2\cos\frac{2\pi}{11} - \epsilon$ for $v \in T^+$ and $a_v(\pi) < -2\cos\frac{2\pi}{11} + \epsilon$ for $v \in T^-$.*

Here $2\cos\frac{2\pi}{11} = 1.68...$

4.5 Weak Ramanujan property.

We can prove that cuspidal representations of GL_4 satisfy the weak Ramanujan property. Let $\pi = \otimes_v \pi_v$ be a cuspidal representation of $GL_4(\mathbb{A})$. Let π_v be an unramified component with the trace a_v, i.e., $a_v = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, where the Satake parameter of π_v is given by $diag(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Then given $\epsilon > 0$, the set of places where $|a_v| \geq q_v^\epsilon$ has density zero.

Note that at a place where π_v is non-tempered, the trace a_v has one of the following three forms: Here u_1, u_2, u_3 are complex numbers with absolute value one. We suppress the dependence of all the factors on v for the simplicity of notation, except a_v.

S_1; $a_v = u_1 q^a + u_2 q^a + u_1 q^{-a} + u_2 q^{-a}$, where $0 < a < \frac{1}{2}$;
S_2; $a_v = u_1 q^a + u_2 + u_3 + u_1 q^{-a}$, where $0 < a < \frac{1}{2}$;
S_3; $a_v = u_1 q^{a_1} + u_2 q^{a_2} + u_1 q^{-a_1} + u_2 q^{-a_2}$, where $0 < a_2 < a_1 < \frac{1}{2}$.

Fix $\epsilon > 0$. Then inside S_2, the set of places where $|a_v| > q^\epsilon$ has density zero. It means the set of places where $a > \epsilon$ has density zero.

Suppose S_1 has a subset S' of positive density where $q^a > q^\epsilon$ for $v \in S'$. Then consider the lift $\wedge^2 \pi$. For $v \in S_1$, the trace of $\wedge^2 \pi_v$ has the form

$$b_v = u_1 u_2 q^{2a} + u_1^2 + u_1 u_2 + u_2^2 + u_1 u_2 q^{-2a}.$$

Then $|b_v| > q^\epsilon$ for $v \in S'$. This is a contradiction.

Suppose S_3 has a subset S' of positive density where $q^{a_1} > q^\epsilon$ for $v \in S'$. Then consider the lift $\wedge^2 \pi$. For $v \in S_3$, the trace of $\wedge^2 \pi_v$ has the form

$$b_v = u_1 u_2 q^{a_1 + a_2} + u_1 u_2 q^{a_1 - a_2} + u_1^2 + u_2^2 + u_1 u_2 q^{-a_1 + a_2} + u_1 u_2 q^{-a_1 - a_2}.$$

Then $|b_v| > q^\epsilon$ for $v \in S'$. This is a contradiction.

4.6 Hypothesis H.

Let $\pi = \otimes \pi_p$ be a cuspidal automorphic representation for $GL_m(\mathbb{A}_\mathbb{Q})$. For p a prime at which π_p is spherical, let $diag(\alpha_{1,p}, ..., \alpha_{m,p})$ be the corresponding Satake parameter. For a positive integer k, let $a_\pi(p^k) = \alpha_{1,p}^k + \cdots + \alpha_{m,p}^k$. Then Rudnick and Sarnak made the following

Hypothesis H. *For any fixed $k \geq 2$,*

$$\sum_p \frac{(\log p)^2 |a_\pi(p^k)|^2}{p^k} < \infty.$$

It has an important application to distribution of zeros of L-functions. Rudnick and Sarnak proved Hypothesis H for $m = 2, 3$. Using the functoriality of the exterior square of GL_4, we can prove Hypothesis H for $m = 4$.

Consider the exterior square $\wedge^2\pi$. There exists an automorphic representation Π of $GL_6(\mathbb{A}_{\mathbb{Q}})$ such that $\Pi_p \simeq \wedge^2\pi_p$ except possibly for $p = 2, 3$, and Π is an isobaric sum of (unitary) cuspidal representations of GL_{n_i}. If π_p is non-tempered, the Satake parameter of $\wedge^2\pi_p$ is as follows:

$S_1 : diag(u_1u_2p^{2a}, u_1u_2, u_1^2, u_2^2, u_1u_2, u_1u_2p^{-2a})$
$S_2 : diag(u_1u_2p^{a}, u_1u_3p^{a}, u_1^2, u_2u_3, u_1u_2p^{-a}, u_1u_3p^{-a})$
$S_3 : diag(u_1u_2p^{a_1+a_2}, u_1u_2p^{a_1-a_2}, u_1^2, u_2^2, u_1u_2p^{-a_1+a_2}, u_1u_2p^{-a_1-a_2})$.

Hence by applying Luo-Rudnick-Sarnak to Π, we see that if $p \in S_1$, $2a \leq \frac{1}{2} - \frac{1}{37}$; if $p \in S_2$, $a \leq \frac{1}{2} - \frac{1}{17}$; if $p \in S_3$, $0 < a_2 < a_1 \leq \frac{1}{2} - \frac{1}{17}$ and $a_1 + a_2 \leq \frac{1}{2} - \frac{1}{37}$.

If $p \in S_1$, $a \leq \frac{1}{4} - \frac{1}{74}$. Hence

$$\sum_{p \in S_1} \frac{(\log p)^2 |a_\pi(p^k)|^2}{p^k} < \infty.$$

If $p \in S_2$, we apply the same technique in Rudnick-Sarnak. Namely, note that $|u_1p^a + u_2 + u_3 + u_1p^{-a}| \geq p^a + p^{-a} - 2$. Hence

$$|\alpha_{1,p}| + |\alpha_{2,p}| + |\alpha_{3,p}| + |\alpha_{4,p}| \leq |\alpha_{1,p} + \alpha_{2,p} + \alpha_{3,p} + \alpha_{4,p}| + 4.$$

Hence

$$|a_\pi(p^k)|^2 << |a_\pi(p)|^{2k}.$$

Since $|a_\pi(p)| << p^{\frac{1}{2} - \frac{1}{17}}$,

$$\sum_{p \in S_2} \frac{(\log p)^2 |a_\pi(p^k)|^2}{p^k} << \sum_{p \in S_2} \frac{(\log p)^2 |a_\pi(p)|^2}{p^{1+\frac{2}{17}(k-1)}}.$$

Since $k > 1$, we apply the fact that $L(s, \pi \times \tilde{\pi})$ converges absolutely for $Re\, s > 1$, and hence

$$\sum_{p \in S_2} \frac{(\log p)^2 |a_\pi(p^k)|^2}{p^k} < \infty.$$

If $p \in S_3$, note that $2a_2 \leq \frac{1}{2} - \frac{1}{37}$. Hence $a_2 \leq \frac{1}{4} - \frac{1}{74}$. Note also that $|u_1(p^{a_1} + p^{-a_1}) + u_2(p^{a_2} + p^{-a_2})| \geq p^{a_1} + p^{-a_1} - (p^{a_2} + p^{-a_2})$. Hence

$$|\alpha_{1,p}| + |\alpha_{2,p}| + |\alpha_{3,p}| + |\alpha_{4,p}| \leq |\alpha_{1,p} + \alpha_{2,p} + \alpha_{3,p} + \alpha_{4,p}| + 2p^{a_2}.$$

Hence

$$|a_\pi(p^k)|^2 << |a_\pi(p)|^{2k} + p^{2ka_2}.$$

We again obtain

$$\sum_{p \in S_3} \frac{(\log p)^2 |a_\pi(p^k)|^2}{p^k} < \infty.$$