WHAT ARE L-FUNCTIONS AND WHAT ARE THEY GOOD FOR?

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The main reference is [1].

1. Lecture One: What are L-functions?

An L-function is a very special type of meromorphic functions of one complex variable. On the surface it is not clear why the L-functions play decisive roles.

An L-function is associated to the set A; arithmetic-geometric objects such as Galois groups, elliptic curves, and Shimura varieties. It is also associated to the set B; automorphic forms and representations. Langlands conjecture is that the set B contains the set A. The L-functions in the set B are called automorphic L-functions. Special case of such relationship for elliptic curves is called Taniyama-Shimura-Wiles theorem, i.e., elliptic curves over $\mathbb{Q}$ are modular.

Over $\mathbb{Q}$, an L-function which is associated to an object F takes the form of Euler product over all primes p, $L(s,F)=\prod_p L_p(s,F)$, $L_p(s,F)=\prod_{j=1}^m (1-\alpha_j(p)p^{-s})^{-1}$ for almost all primes, where $\alpha_j(p)\in\mathbb{C}$. As a function of $s\in\mathbb{C}$, this product converges absolutely for $Re(s)>>0$ and we can multiply out to get a series $L(s,F)=\sum a(n)n^{-s}$.

Examples: (1) Riemann zeta function $\zeta(s)=\prod_p(1-p^{-s})^{-1}=\sum_{n=1}^\infty n^{-s}$. It was Riemann who introduced his zeta function in order to study the distribution of prime numbers: The prime number theorem says that $\pi(x)=\sum_{p\le x}1\sim\frac{x}{\log x}$. It is equivalent to the fact that $\zeta(1+it)\neq 0$ for $t\in\mathbb{R}$. A better zero free region gives a better error term. Riemann hypothesis is $\zeta(s)\neq 0$ if $Re(s)>\frac{1}{2}$. It gives $\pi(x)=Li(x)+O(\sqrt{x}\log x)$, where $Li(x)=\int_2^x\frac{dt}{\log t}$.

(2) Dirichlet L-function $L(s,\chi)=\prod_p(1-\chi(p)p^{-s})^{-1}=\sum_{n=1}^\infty\chi(n)n^{-s}$, where χ is a character of $(\mathbb{Z}/q\mathbb{Z})^\times$.

Dirichlet's theorem on arithmetic progression says that there are infinitely many primes in the arithmetic progression $an+b$, where $(a,b)=1$, $n=1,2,...$ It comes from the fact that $L(1,\chi)\neq 0$ for a Dirichlet character χ mod a. A better zero free region gives rise to a better error term.

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(3) $E : y^2 = x^3 + ax + b, a, b \in \mathbb{Z}$. Let $N_E(p)$ be the number of solutions mod p. Let $a_E(p) = p - N_E(p)$. Then $L_p(s, E) = (1 - a_E(p)p^{-s} + p^{1-2s})^{-1}$ for p non-singular. (Without normalization: In order to have a functional equation of the form $s \longmapsto 1 - s$, we need to take $L_p(s, E) = (1 - a_E(p)p^{-\frac{1}{2}-s} + p^{-2s})^{-1}$.)

More explicitly, let $E : y^2 = x^3 - 4x^2 + 16$. Then we have some numerical calculations (due to Silverman):

p	2	3	5	7	11	13	17	19	23
$N_E(p)$	2	4	4	9	10	9	19	19	24
$a_E(p)$	0	-1	1	-2	1	4	-2	0	-1

Let $F = q\prod_{k=1}^{\infty}[(1 - q^k)(1 - q^{11k})]^2 = \sum_{n=1}^{\infty} b_n q^n$. Here F is a modular form of weight 2. Consider the L-function $L(s, F) = \sum_{n=1}^{\infty} b_n n^{-s}$. Then

$$L(s, F) = \prod_{p \neq 11} (1 - b_p p^{-s} + p^{1-2s})^{-1}(1 - b_{11}11^{-s})^{-1}.$$

We have $a_E(p) = b_p$ for all $p \neq 2$.

(4) Artin L-functions. Let L/K be a Galois extension and let $P(L/K)$ be the set of prime ideals in K which split completely in L. Fact $P(L/K)$ determines L completely.

The goal of class field theory is to describe the Galois extension L in terms of data in K, namely, determine $P(L/K)$ in terms of data in K. When L/K is abelian, the answer is given completely by the class field theory. For example, $P(\mathbb{Q}(\zeta_m)/\mathbb{Q}) = \{p \equiv 1 \pmod m\}$, where ζ_m is a primitive mth roots of unity. But when L/K is not abelian, not much is known.

Example 1 (cf. [13]). Consider $f(x) = x^3 - x - 1$. Here the discriminant is -23. Let L be the splitting field of f. Then $Gal(L/\mathbb{Q}) \simeq S_3$. If p is unramified, $f(x) \equiv 0 \pmod p$ has 0,1,3 solutions. Then $P(L/\mathbb{Q}) = \{p|\, f(x) \equiv 0 \pmod p \text{ has 3 solutions.}\}$. By computer calculation, we see that $P(L/\mathbb{Q}) = \{59, 101, 167, 173, ...\}$. On the other hand, $f(x) \equiv 0 \pmod p$ has 0 solutions when $p = 2, 3, 13, 29, 31, 41, ...$ It is hard to see the pattern. The pattern comes from modular forms. Let $\rho : S_3 \longrightarrow GL_2(\mathbb{C})$ be the 2-dimensional representation of S_3. Then we have the Artin L-function $L(s, \rho, L/\mathbb{Q})$. It is given by the Euler product

$$L(s, \rho, L/\mathbb{Q}) = \prod_p (1 - a_p p^{-s} + \left(\frac{p}{23}\right) p^{-2s})^{-1},$$

where $a_p = N_f(p) - 1$, and $N_f(p)$ is the number of solutions for $f(x) \equiv 0 \pmod p$. Here $L(s, \rho, L/\mathbb{Q}) = \frac{\zeta_E(s)}{\zeta(s)}$, where $E = \mathbb{Q}(\alpha)$, and α is a root of $f(x)$. This comes from the fact that $Ind_H^{S_3} 1 = 1 + \rho$. Here H is the Galois group of L/E, and $H \simeq \mathbb{Z}/2\mathbb{Z}$.

Theorem 1.1 (Langlands functoriality)**.** *(1) $L(s,\rho,L/\mathbb{Q})$ is the L-function attached to a modular form of weight 1, level 23, with respect to the character $\epsilon(p)=\left(\frac{p}{23}\right)$. More precisely, $L(s,\rho,L/\mathbb{Q})=L(s,F)$,*

$$F=q\prod_{k=1}^{\infty}(1-q^k)(1-q^{23k})=\sum_{n=1}^{\infty}a_nq^n,\quad q=e^{2\pi i\tau},\tau=x+iy$$
$$L(s,F)=\prod_p(1-a_pp^{-s}+\left(\frac{p}{23}\right)p^{-2s})^{-1}.$$

(2) $P(L/\mathbb{Q})=\{p|\, a_p=2\}$.

Example 2. $f(x)=x^5-x-1$. The Galois group is S_5. It is not solvable. Let L be the splitting field of f and $E=\mathbb{Q}(\alpha)$, where α is a root of f. Let $H=Gal(L/E)$. Then $H\simeq S_4$, and $Ind_H^{S_5}1=1+\rho$, where $\rho:S_5\longrightarrow GL_4(\mathbb{C})$ be a 4-dimensional representation. We have the Artin L-function $L(s,\rho,L/\mathbb{Q})=\frac{\zeta_E(s)}{\zeta(s)}$. The strong Artin conjecture says that there exists a cuspidal representation π of $GL_4/\mathbb{Q}$ such that $L(s,\rho,L/\mathbb{Q})=L(s,\pi)$. It has been proved recently by Khare and others [3].

Conjecture 1.2 (Langlands functoriality conjecture)**.** *There exists a cuspidal representation $\pi=\otimes\pi_p$ of GL_4 such that $L(s,\rho,L/\mathbb{Q})=L(s,\pi)$, and $P(L/\mathbb{Q})=\{p|$ Satake parameter of π_p is $diag(1,1,1,1)\}$.*

A weaker assertion is the Artin conjecture: $L(s,\rho,L/\mathbb{Q})$ is entire.

We only know that $L(s,\rho,L/\mathbb{Q})$ has meromorphic continuation to all of $\mathbb{C}$ and satisfies a functional equation.

More generally, Langlands conjectured that given an irreducible representation $\rho:Gal(\overline{\mathbb{Q}}/\mathbb{Q})\longrightarrow GL_n(\mathbb{C})$, there exists a cuspidal representation $\pi=\otimes\pi_p$ of GL_n such that $\rho(Frob_p)=$Satake parameter of π_p. It is usually referred to as the strong Artin conjecture.

Much effort has been made when $n=2$. Let $\bar\rho:Gal(\overline{\mathbb{Q}}/\mathbb{Q})\longrightarrow PGL_2(\mathbb{C})\simeq SO_3(\mathbb{C})$. Then $Im(\bar\rho)$ is D_{2n} (dihedral), A_4 (tetrahedral), S_4 (octahedral), A_5 (icosahedral). The first three groups are solvable. It is a theorem of Langlands and Tunnell that for the case of A_4,S_4, the strong Artin conjecture is true, and it has been used by Andrew Wiles in his proof of Fermat's last theorem.

Cuspidal representations generalize classical modular forms (holomorphic and Maass forms). They can be understood as "direct summands" of the right regular representation of $G(\mathbb{A})$ on the Hilbert space $L^2(G(F)\backslash G(\mathbb{A}))$, where $\mathbb{A}$ is the ring of adeles. If π is a cuspidal representation, then we have a tensor product decomposition $\pi = \otimes \pi_v$, where v runs through all places of F; π_v is an irreducible, unitary representation of $G(F_v)$ for all v; π_v is spherical for almost all v, namely, it has the Satake parameter (semi-simple conjugacy class) $\{t_v\}$ in ${}^LG$ (the L-group of G).

When $G = GL_2$, $F = \mathbb{Q}$, there are two types of cuspidal representations: First, cuspidal representations attached to holomorphic cusp forms of weight k with respect to a congruence subgroup of $SL_2(\mathbb{Z})$; $\pi = \pi_f$, where $f(\tau) = \sum_{n=1}^{\infty} a_n n^{\frac{k-1}{2}} e^{2\pi i n\tau}, \tau = x + iy$. Then $\pi_f = \otimes\pi_p$, and the Satake parameter of π_p is $diag(\alpha_p, \beta_p)$, where $a_1 = 1, a_p = \alpha_p + \beta_p$.

Second, cuspidal representations attached to Maass cusp forms. They are eigenfunctions of the Laplacian; $y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f + (\frac{1}{4} - t^2)f = 0$, where $t \in i\mathbb{R}$ or $t \in \mathbb{R}$, $0 < t < \frac{1}{2}$. Then $f(\tau) = \sum_{n\neq 0} a_n |n|^{-\frac{1}{2}} W(n\tau)$, where $W(\tau) = y^{\frac{1}{2}} K_t(2\pi y) e^{2\pi i x}$ and K_t is the K-Bessel function.

Conjecture 1.3 (Ramanujan conjecture)**.** $|\alpha_p| = |\beta_p| = 1$.

Conjecture 1.4 (Selberg conjecture)**.** *$t \in i\mathbb{R}$, or $\frac{1}{4} - t^2 \geq \frac{1}{4}$.*

Theorem 1.5 (Deligne, 1973))**.** *Ramanujan conjecture is true for holomorphic cusp forms.*

(5) Symmetric power L-functions. Let $Sym^m : GL_2(\mathbb{C}) \longrightarrow GL_{m+1}(\mathbb{C})$ be the mth symmetric power representation on the space of homogeneous polynomials of 2 variables of degree m. For $g \in GL_2(\mathbb{C})$, $g \cdot f(x, y) = f(X, Y)$, where $\begin{pmatrix} X \\ Y \end{pmatrix} = g \begin{pmatrix} x \\ y \end{pmatrix}$ and f is a homogeneous polynomial of degree m.

Let $\pi = \otimes\pi_p$ be a cuspidal representation of $GL_2(\mathbb{A})$ such that $diag(\alpha_p, \beta_p)$ be the Satake parameter of π_p for almost all p. Then

$$L(s, \pi_p, Sym^m) = \prod_{j=0}^{m} (1 - \alpha_p^{m-j} \beta_p^j p^{-s})^{-1}.$$

The L-function $L(s, \pi, Sym^m)$ was introduced by Langlands to solve Ramanujan and Sato-Tate conjecture. For example, if we know that $L(s, \pi, Sym^m)$ is absolutely convergent for $Re(s) > 1$ for all m, then $|\alpha_p^m| \leq p, |\beta_p^m| \leq p$ for all m. This implies that $|\alpha_p| \leq p^{\frac{1}{m}}, |\beta_p| \leq p^{\frac{1}{m}}$ for all m. Since $|\alpha_p\beta_p| = 1$, we have $|\alpha_p| = |\beta_p| = 1$.

Sato-Tate conjecture (now a theorem due to Taylor and et al): Let π be a cuspidal representation with the trivial central character. Let $a_p = \alpha_p + \alpha_p^{-1} = 2\cos\theta_p$, $0 \leq \theta_p \leq \pi$. Then for

$0 \le a < b \le \pi$,

$$\frac{1}{\pi(x)}\#\{p \le x : \theta_p \in (a,b)\} \to \frac{2}{\pi}\int_a^b \sin^2 d\theta, \quad x \to \infty.$$

Serre showed that the analytic continuation of $L(s,\pi,Sym^m)$ and non-vanishing for $Re(s) \ge 1$ imply Sato-Tate conjecture. Now it ia a theorem [11] that $L(s,\pi,Sym^m)$ is modular for π coming from holomorphic modular forms.

2. Lecture Two: How do we study L-functions?

Let $\pi = \otimes \pi_v$ be a cuspidal representation of $G(\mathbb{A})$, where G is a split reductive group. Let ${}^LG$ be the L-group of G. Let $r : {}^LG \longrightarrow GL_N(\mathbb{C})$ be a finite dimensional representation of ${}^LG$. For $v \notin S$, π_v is spherical and it gives rise to a Satake parameter (semi-simple conjugacy class) $\{t_v\}$, $t_v \in {}^LG$. Form the local L-function $L(s,\pi_v,r) = det(I - r(t_v)q_v^{-s})^{-1}$. Let $L_S(s,\pi,r) = \prod_{v \notin S} L(s,\pi_v,r)$.

Central problems: (1) $L(s,F)$ has a meromorphic continuation to all of $\mathbb{C}$ and satisfies a function equation of the form: Let $\Lambda(s,F) = L(s,F)\times$ (some γ-factors and factors at bad places). Then $\Lambda(s,F) = \epsilon(s,F)\Lambda(1-s,F')$, where F' is an object related to F such as a contragredient representation. For example, $\Lambda(s) = \zeta(s)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) = \Lambda(1-s)$.

(2) $\Lambda(s,F)$ is bounded in vertical strips.

(3) Grand Riemann Hypothesis; non-trivial zeros of $L(s,F)$ are all on $Re(s) = \frac{1}{2}$.

(4) Generalized Ramanujan conjecture: $|\alpha_j(p)| = 1$.

(5) Birch, Swinnerton-Dyer conjecture: Let $E/\mathbb{Q}$ be an elliptic curve. The order of vanishing of $L(s,E)$ at $s = 1$ (center of symmetry) is equal to the rank of the group of rational points on E.

(6) Other problems such as Siegel zeros (real zeros close to 1). For example, the formula for $L(1,\chi)$ contains the class number of $K/\mathbb{Q}$, where K is a quadratic extension, and χ quadratic character of $K/\mathbb{Q}$. The absence of Siegel zeros gives strong result on class number. For example, a Siegel zero for $K = \mathbb{Q}(\sqrt{-D})$, $D > 0$, is a zero between $(1 - \frac{a}{\log D}, 1)$. Absence of Siegel zeros implies that $h(D) \gg \frac{\sqrt{D}}{\log D}$ with effective constant.

Here (3) and (5) are two of seven one million dollar prize problems of Clay Math. Institute.

Here even meromorphic continuation is not obvious. For example, it is clear that $\prod_{p\equiv 1 \pmod 4}(1-p^{-s})^{-1}$ converges for $Re(s) > 1$. We can continue up to $Re(s) > 0$. But it is known that it has no meromorphic continuation to all of $\mathbb{C}$. It has a natural boundary at $Re(s) = 0$ (Kurokawa).

There are two ways of studying automorphic L-functions:

(1) method of Rankin-Selberg (integral representations); expresses L-functions as integrals of Eisenstein series, theta functions, etc. For example, Riemann proved that

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\int_0^\infty x^{\frac{s}{2}}\frac{1}{2}(\theta(x)-1)\,\frac{dx}{x},\quad \theta(x)=\sum_{-\infty}^{\infty}e^{-n^2\pi x}.$$

The Poisson summation formula gives rise to the functional equation of the theta function, $\theta(x^{-1})=x^{\frac{1}{2}}\theta(x)$, and the functional equation of the Riemann zeta function follows.

Let π,π' be cuspidal representations of GL_n, and $E(s,g)$ be the Eisenstein series of GL_n associated with maximal parabolic subgroup with the Levi subgroup $GL_{n-1}\times GL_1$. Let ϕ,ϕ' be automorphic forms in the space of π,π', resp. Then

$$I(s,\phi,\phi')=\int_{Z_nGL_n(\mathbb{Q})\backslash GL_n(\mathbb{A})}\phi(g)\phi'(g)E(s,g)\,dg=L(s,\pi\times\pi')\times \text{bad factor}.$$

We can study $L(s,\pi\times\pi')$ using this integral representation.

(2) Langlands-Shahidi method; uses Eisenstein series attached to maximal parabolic subgroups.

Let $P=MN$ be a maximal parabolic subgroup of G. Let π be a cuspidal representation of $M(\mathbb{A})$. Then we can form an induced representation, for $s\in\mathbb{C}$,

$$I(s,\pi)=Ind_P^G\pi\otimes exp(s\tilde{\alpha},H_P(\,)),$$

where $\tilde{\alpha}$ is the fundamental weight corresponding to α, and α is a simple root such that P is associated to $\Delta-\{\alpha\}$. (Δ is the set of simple roots) For example, if $P=MN\subset Sp_{2n}$, $M\simeq GL_n$ (Siegel parabolic subgroup), then $I(s,\pi)=Ind_P^G\pi\otimes|det|^s$.

Given $f_s\in I(s,\pi)$, we define an Eisenstein series

$$E(s,f_s,g)=\sum_{\gamma\in P(F)\backslash G(F)}f_s(\gamma g).$$

Let $E_0(s,f_s,g)=\int_{N(F)\backslash N(\mathbb{A})}E(s,f_s,ng)\,dn$. It is called constant term. If P is self-conjugate, i.e., $w_0(\Delta-\{\alpha\})=\Delta-\{\alpha\}$ (most cases),

$$E_0(s,f_s,g)=f_s(g)+M(s,\pi)f_s(g),\quad M(s,\pi)f_s(g)=\int_{N(\mathbb{A})}f_s(w_0^{-1}ng)\,dn,$$

where w_0 is a Weyl group element. $M(s,\pi)$ is called global intertwining operator from $I(s,\pi)$ to $I(-s,w_0(\pi))$.

Langlands [10] proved that the poles of $E(s,f_s,g)$ and $M(s,\pi)$ are the same and they have meromorphic continuation to all of $\mathbb{C}$ and satisfy a functional equation $E(-s,M(s,\pi)f_s,g)=E(s,f_s,g)$.

Let $f_s = \otimes f_v$ and $I(s,\pi) = \otimes I(s,\pi_v)$. Then $M(s,\pi) = \otimes A(s,\pi_v,w_0)$, where

$$A(s,\pi_v,w_0)f_v(g) = \int_{N(F_v)} f_v(w_0^{-1}ng)\,dn.$$

It is called local intertwining operator. For almost all v, f_v is the unique K_v-fixed vector in $I(s,\pi_v)$ such that $f_v(e) = 1$. Then Langlands proved [9]

$$A(s,\pi_v,w_0)f_v = \prod_{i=1}^m \frac{L(is,\pi_v,r_i)}{L(1+is,\pi_v,r_i)}\tilde{f}_v,$$

[Gindikin-Karpelevich formula], where $\tilde{f}_v$ is the unique K_v-fixed vector in $I(-s,w_0(\pi_v))$ and r_i is certain irreducible finite-dimensional representation of ${}^LM$. Hence

$$M(s,\pi) = \prod_{i=1}^m \frac{L_S(is,\pi,r_i)}{L_S(1+is,\pi,r_i)} \otimes_{v\notin S} \tilde{f}_v \otimes \otimes_{v\in S} A(s,\pi_v,w_0)f_v.$$

We can show that for $i>1$, $L(s,\pi,r_i) = L(s,\pi',r_1')$ for some π' on $M'\subset G'$. By induction on i, this gives a meromorphic continuation of $L_S(s,\pi,r_i)$ for each i. But it does not give the desired functional equation.

At the suggestion of Langlands, Shahidi [14] calculated ψ-Fourier coefficients of $E(s,f_s,g)$ for globally generic cuspidal representations, where ψ is a generic character of U. Here U is a maximal unipotent subgroup such that $B = TU$ is a Borel subgroup. Then $\psi_M = \psi|_{U_M}$ is a generic character of $U_M = U\cap M$. We say that $\pi = \otimes\pi_v$ is ψ_M-globally generic if $\int_{U_M(F)\backslash U_M(\mathbb{A})} \varphi(ug)\overline{\psi_M(u)}\,du \neq 0$ for a cuspidal function φ in the space of π. This implies that each π_v is locally generic, i.e., has a Whittaker model. If $\lambda_{\psi_v}(s,\pi_v)$ is the Whittaker functional for the space of $I(s,\pi_v)$, then by the uniqueness of Whittaker functional up to a constant,

$$\lambda_{\psi_v}(s,\pi_v) = C_{\psi_v}(s,\pi_v,w_0)\lambda_{\psi_v}(-s,w_0(\pi_v))A(s,\pi_v,w_0),$$

for some constant $C_{\psi_v}(s,\pi_v,w_0)\in\mathbb{C}$.

Consider ψ-Fourier coefficient of Eisenstein series

$$E_\psi(s,f_s,g) = \int_{U(F)\backslash U(\mathbb{A})} E(s,f_s,ug)\overline{\psi(u)}\,du.$$

Then Shahidi showed that

$$E_\psi(s,f_s,e) = \frac{\prod_{v\in S} W_v(e_v)}{\prod_{i=1}^m L_S(1+is,\pi,r_i)}, \quad W_v(e_v) = \lambda_{\psi_v}(s,\pi_v)(f_v).$$

Functional equation of Eisenstein series implies

$$\prod_{i=1}^{m} L_s(is,\pi,r_i) = \prod_{v\in S} C_{\psi_v}(s,\pi_v,w_0)\prod_{i=1}^{m} L_S(1-is,\pi,\tilde{r}_i).$$

Induction on i and detailed analysis of $C_{\psi_v}(s,\pi_v,w_0)$ lead to the definition of local factors $L(s,\pi_v,r_i),\epsilon(s,\pi_v,r_i,\psi_v)$ for $v\in S$, and the functional equation $L(s,\pi,r_i)=\epsilon(s,\pi,r_i)L(1-s,\pi,\tilde{r}_i)$ (Shahidi 1990).

But it had been thought that the location of poles is hard to obtain from Langlands-Shahidi method. Things changed by using spectral theory:

Theorem 2.1 (Langlands' Lemma)**.** [6] *Consider the residual spectrum* $L^2=L^2_{res}(G(F)\backslash G(\mathbb{A}))_{(M,\pi)}$.

- $L^2=\{0\}$ *unless* $w_0\pi\simeq\pi$.
- L^2 *is spanned by the residues of* $E(s,f_s,g)$ *for* $s>0$.
- *Suppose* $E(s,f_s,g)$ *has a pole at* $s=s_0$. *Then* $L^2=\{\otimes_v J(s,\pi_v)\}$, *where* $J(s,\pi_v)$ *is the image of* $N(s,\pi_v,w_0)$, *the normalized local intertwining operator, namely,*

$$A(s,\pi_v,w_0)=\prod_{i=1}^{m}\frac{L(is,\pi_v,r_i)}{L(1+is,\pi_v,r_i)\epsilon(is,\pi_v,r_i)}N(s,\pi_v,w_0).$$

Then

$$M(s,\pi)=\prod_{i=1}^{m}\frac{L(is,\pi,r_i)}{L(1+is,\pi,r_i)\epsilon(is,\pi,r_i)}\otimes_v N(s,\pi_v,w_0).$$

(1) and (2) imply that $M(s,\pi)$ is holomorphic for $s>0$ unless $w_0\pi\simeq\pi$.

Trick: $w_0(\pi\otimes\chi)\not\simeq\pi\otimes\chi$ if χ is a grössencharacter which is highly ramified at one finite place.

We denote $\pi\otimes\chi$ by π. Then $M(s,\pi)$ is holomorphic for $s>0$.

We have to show that each local operator $N(s,\pi_v,w_0)$ is holomorphic and non-vanishing for $Re(s)\geq\frac{1}{2}$. This requires the study of representations of p-adic groups. The main ingredients are the following standard module conjecture and classification of discrete series representations.

Theorem 2.2 (Standard module conjecture)**.** *Given a non-tempered generic representations* π_v, *there is a tempered data* π_0 *and a complex parameter* Λ_0 *which is in the corresponding positive Weyl chamber so that* $\pi_v=I_{M_0}(\Lambda_0,\pi_0)=Ind_{M_0}^{M}(\pi_0\otimes q_v^{<\Lambda_0,H_{P_0}^{M}(\ )>})$.

According to Langlands, any non-tempered representation π_v can be written as a Langlands' quotient, namely, the quotient of $I_{M_0}(\Lambda_0,\pi_0)$. The above conjecture claims that if π_v is generic, π_v is $I_{M_0}(\Lambda_0,\pi_0)$ itself.

Theorem 2.3. *The local normalized intertwining operators $N(s, \pi_v, w_0)$ are holomorphic and non-vanishing for $Re(s) \geq \frac{1}{2}$.*

2.1. **Langlands' functoriality conjecture.** Let H, G be two reductive groups. To each homomorphism of L-groups, $r : {}^LH \longrightarrow {}^LG$, there is associated a lift (transfer) of automorphic representations of H to automorphic representations of G which satisfy canonical properties.

Example 1. $H = \{e\}$, $G = GL_n$ over $\mathbb{Q}$. Then ${}^LH = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ and ${}^LG = GL_n(\mathbb{C})$. Then Langlands functoriality conjecture is the strong Artin conjecture.

Example 2. $H = GL_2, G = GL_{m+1}$. Let $Sym^m : GL_2(\mathbb{C}) \longrightarrow GL_{m+1}(\mathbb{C})$ be the mth symmetric power representation. Then for π, a cuspidal representation of GL_2, we have an automorphic representation $Sym^m(\pi)$ of GL_{m+1}. If $\pi = \pi_f$, where f is a holomorphic cusp form, then it is a theorem due to Taylor, Thorne,... For a general π, it is only proved for $m \leq 4$; Gelbart-Jacquet ($m = 2$), Kim-Shahidi ($m = 3$), and Kim ($m = 4$).

Example 3. $H = SO_{2n+1}, SO_{2n}, Sp_{2n}$, $G = GL_N$, where $N = 2n$ or $2n + 1$. Then ${}^LH = Sp_{2n}(\mathbb{C}), SO_{2n}(\mathbb{C}), SO_{2n+1}(\mathbb{C})$, and $r : {}^LH \longrightarrow {}^LG$ is the embedding. Arthur trace formula gives the functoriality.

Example 4. Let $SL_2(\mathbb{C}) \times Sp_{2n}(\mathbb{C}) \longrightarrow Spin(4n + 1, \mathbb{C})$ be the embedding. Here ${}^LPGL_2 = SL_2(\mathbb{C})$ and ${}^LPGSp_{4n} = Spin(4n + 1, \mathbb{C})$. So we have a functoriality $(\pi_f, \Pi) \mapsto$ Ikeda lift. Here Π is an anomalous representation $\Pi = \Pi_\infty \otimes \otimes_p \Pi_p$, where Π_∞ is a discrete series representation of $SO_{2n+1}(\mathbb{R})$, and Π_p is the quotient of $Ind_B^{SO_{2n+1}} |\ |^{\frac{2n-1}{2}} \otimes |\ |^{\frac{2n-3}{2}} \otimes \cdots \otimes |\ |^{\frac{1}{2}}$.

We can use the converse theorem of Cogdell-Piatetski-Shapiro to prove some cases of functoriality:

Theorem 2.4 (Converse theorem)**.** *Suppose $\Pi = \otimes \Pi_v$ is an irreducible, admissible representation of GL_N such that $\omega_\Pi = \otimes \omega_{\Pi_v}$ is a grössencharacter. Let S be a finite set of finite places and let $\mathcal{T}^S(m)$ be the set of all cuspidal representations of GL_m that are unramified at S. Suppose $L(s, \sigma \times \Pi)$ is nice (entire, functional equation, bounded in vertical strips) for all $\sigma \in \mathcal{T}^S(m) \otimes \chi$, $m = 1, ..., N - 2$, where χ is a grössencharacter which is highly ramified at S. Then there exists an automorphic representation Π' of GL_N such that $\Pi'_v \simeq \Pi_v$ for $v \notin S$.*

Outline of the proof of functoriality of $Sym^3(\pi)$:

This is obtained indirectly from the functorial product associated with the tensor product map

$$GL_2(\mathbb{C}) \times GL_3(\mathbb{C}) \longrightarrow GL_6(\mathbb{C}).$$

Let π_1, π_2 be cuspidal representations of GL_2, GL_3, resp. For each place v, π_{iv}, $i = 1, 2$, are parametrized by $\phi_{iv} : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow GL_{i+1}(\mathbb{C})$, $i = 1, 2$. Then $\phi_{1v} \otimes \phi_{2v} : W_{F_v} \times SL_2(\mathbb{C}) \longrightarrow GL_6(\mathbb{C})$. By the local Langlands correspondence, $\phi_{1v} \otimes \phi_{2v}$ gives rise to an irreducible, admissible representation $\pi_{1v} \boxtimes \pi_{2v}$ of $GL_6(F_v)$. Let $\pi_1 \boxtimes \pi_2 = \otimes(\pi_{1v} \boxtimes \pi_{2v})$. It is an irreducible, admissible representation of $GL_6(\mathbb{A})$.

Theorem 2.5 (Kim-Shahidi)**.** [8] $\pi_1 \boxtimes \pi_2$ *is automorphic.*

Let π be a cuspidal representation of GL_2, and let $Ad(\pi)$ be the adjoint square, i.e., $Ad(\pi) = Sym^2(\pi) \otimes \omega_\pi^{-1}$, where ω_π is the central character. Then $\pi \boxtimes Ad(\pi) = (Sym^3(\pi) \otimes \omega_\pi^{-1}) \boxplus \pi$. Here $\boxplus$ is the isobaric sum, and it denotes the unitary induction $Ind_{GL_4 \times GL_2}^{GL_6}(Sym^3(\pi) \otimes \omega_\pi^{-1}) \otimes \pi$.

Apply the converse theorem to $\pi_1 \boxtimes \pi_2$. Let S be a finite set of finite places such that π_{1v}, π_{2v} are spherical for $v \notin S$, $v < \infty$. We need the triple product L-functions

$$L(s, \sigma \times (\pi_1 \boxtimes \pi_2)) = L(s, \sigma \times \pi_1 \times \pi_2),$$

where σ is a cuspidal representation of GL_m, $m = 1, 2, 3, 4$. These are available from Langlands-Shahidi method:

- $m = 1$: Rankin-Selberg L-function of $GL_2 \times GL_3$
- $m = 2$: $D_5 - 2$ case. Use $Spin(10)$.
- $m = 3$: $E_6 - 1$ case. Use simply connected E_6.
- $m = 4$: $E_7 - 1$ case. Use simply connected E_7.

Functional equation: due to Shahidi (1990)

Bounded in vertical strips: due to Gelbart and Shahidi (2001)

Entire: trick is to use χ which is highly ramified at one finite place

3. Arithmetic properties of L-functions

3.1. **Sato-Tate, vertical Sato-Tate, and central limit theorem.** Let $\mathcal{F}_k$ be the set of Hecke eigen new forms of weight k with respect to $SL_2(\mathbb{Z})$. Let $\pi = \pi_f$, $f \in \mathcal{F}_k$, be the cuspidal representation of GL_2. Let $L(s, \pi_f) = \sum_{n=1}^\infty a_f(n) n^{-s}$, where $a_f(p) = 2\cos\theta_f(p)$. Then we have

Sato-Tate: for any continuous function $h : [-2, 2] \longrightarrow \mathbb{R}$,

$$\frac{1}{\pi(x)} \sum_{p \le x} h(a_f(p)) \longrightarrow \frac{1}{2\pi} \int_{-2}^{2} h(t)\sqrt{4 - t^2}\, dt, \quad x \to \infty.$$

Vertical Sato-Tate: Fix p. For any continuous function $h : [-2,2] \longrightarrow \mathbb{R}$,

$$\frac{1}{|\mathcal{F}_k|}\sum_{f\in\mathcal{F}_k} h(a_f(p)) \longrightarrow \int_{-2}^{2} h(t)\,\mu_p, \quad k\to\infty,$$

where $\mu_p = \frac{1}{2\pi}(1+\frac{1}{p})\frac{\sqrt{4-t^2}}{(1+\frac{1}{p})^2-\frac{t^2}{p}}\,dt$. It is due to Serre, Conrey-Duke.

Central limit theorem (Nagoshi): For any continuous function $h : \mathbb{R} \longrightarrow \mathbb{R}$, under the condition $\frac{\log k}{\log x} \to \infty$,

$$\frac{1}{|\mathcal{F}_k|}\sum_{f\in\mathcal{F}_k} h\left(\frac{\sum_{p\le x} a_f(p)}{\sqrt{\pi(x)}}\right) \longrightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(t)e^{-\frac{t^2}{2}}\,dt, \quad x\to\infty.$$

We can prove vertical Sato-Tate theorem and central limit theorem for Siegel modular forms of degree 2.

Let $\underline{k} = (k_1,k_2)$, $k_1 \ge k_2 \ge 3$, and $S_{\underline{k}}(\Gamma(N))^{\mathrm{tm}}$ be the space of Siegel holomorphic modular forms of weight $\underline{k}$ which satisfy the Ramanujan conjecture. Let $HE_{\underline{k}}(\Gamma(N))^{\mathrm{tm}}$ be a basis. For $F \in HE_{\underline{k}}(\Gamma(N))^{\mathrm{tm}}$, let $\pi = \pi_F = \otimes\pi_{F,p}$ be the cuspidal representation of GSp_4. Then $\pi_{F,p}$ is tempered, and the Satake parameter is $\{\alpha_{0p}, \alpha_{0p}\alpha_{1p}, \alpha_{0p}\alpha_{2p}, \alpha_{0p}\alpha_{1p}\alpha_{2p}\}$. We write it as $\{\alpha_{F,p}^{\pm}, \beta_{F,p}^{\pm}\}$ such that $a_{F,p} = \alpha_{F,p} + \alpha_{F,p}^{-1} = 2\cos\theta_{1p}$ and $b_{F,p} = \beta_{F,p} + \beta_{F,p}^{-1} = 2\cos\theta_{2p}$, and $\theta_{1p}, \theta_{2p} \in [0,\pi]$. Then

Theorem 3.1 (Vertical Sato-Tate; K-Wakatsuki-Yamauchi)**.** *For a continuous function* $h : [-2,2]^2/S_2 = \Omega \longrightarrow \mathbb{R}$,

$$\frac{1}{|HE_{\underline{k}^{\mathrm{tm}}}|}\sum_{F\in HE_{\underline{k}}^{\mathrm{tm}}} h(a_{F,p}, b_{F,p}) \longrightarrow \int_\Omega h(x,y)\,\mu_p, \quad N+k_1+k_2\to\infty,$$

where

$$\mu_p = \frac{(p+1)^4}{4p^4}\cdot\frac{1}{\pi^2}\left|\frac{(1-e^{2i\theta_1})(1-e^{2i\theta_2})(1-e^{i(\theta_1-\theta_2)})(1-e^{i(\theta_1+\theta_2)})}{(1-p^{-1}e^{2i\theta_1})(1-p^{-1}e^{2i\theta_2})(1-p^{-1}e^{i(\theta_1-\theta_2)})(1-p^{-1}e^{i(\theta_1+\theta_2)})}\right|^2 d\theta_1 d\theta_2.$$

Sato-Tate conjecture is

$$\frac{1}{\pi(x)}\sum_{p\le x} h(a_{F,p}, b_{F,p}) \longrightarrow \int_\Omega h(x,y)\,\mu_\infty^{ST}, \quad x\to\infty,$$

where

$$\begin{aligned}\mu_\infty^{ST} &= \lim_{p\to\infty}\mu_p = \frac{1}{4\pi^2}\left|(1-e^{2i\theta_1})(1-e^{2i\theta_2})(1-e^{i(\theta_1-\theta_2)})(1-e^{i(\theta_1+\theta_2)})\right|^2 d\theta_1 d\theta_2\\ &= \frac{(x-y)^2}{4\pi^2}\sqrt{4-x^2}\sqrt{4-y^2}\,dxdy.\end{aligned}$$

In order to prove this conjecture, we need to prove the holomorphy and non-vanishing of all $L(s,\pi_F,r)$ for $Re(s)\geq 1$ for any finite dimensional representation $r: GSp_4(\mathbb{C})\longrightarrow GL_N(\mathbb{C})$.

Let $L(s,\pi_F,Spin)=\sum_{n=1}^\infty \lambda_F(n)n^{-s}$ be the spinor L-function, where $\lambda_F(p)=a_{F,p}+b_{F,p}$. Then we have

Theorem 3.2 (Central limit theorem; K-Wakatsuki-Yamauchi)**.** *Under the condition* $\frac{\log(N+k_1+k_2)}{\log x}\to\infty$*,*

$$\frac{1}{|HE_{\underline{k}}^{\rm tm}|}\sum_{F\in HE_{\underline{k}}^{\rm tm}} h\left(\frac{\sum_{p\leq x}\lambda_F(p)}{\sqrt{\pi(x)}}\right)\longrightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty h(t)e^{-\frac{t^2}{2}}\,dt,\quad x\to\infty.$$

3.2. **Low lying zeros.** Katz and Sarnak proposed a conjecture on low-lying zeros of L-functions in natural families, which says that the distributions of the low-lying zeros of L-functions in a family $\mathfrak{F}$ is predicted by a symmetry type $G(\mathfrak{F})$ attached to $\mathfrak{F}$: For a given entire L-function $L(s,\pi)$, we denote the non-trivial zeros of $L(s,\pi)$ by $\frac12+\sqrt{-1}\gamma_j$. Since we don't assume GRH for $L(s,\pi)$, γ_j can be a complex number. Let $\phi(x)$ be an even Schwartz class function whose Fourier transform

$$\hat\phi(x)=\int_{-\infty}^\infty \phi(y)e^{-2\pi ixy}\,dy$$

is compactly supported. We define

$$D(\pi,\phi)=\sum_{\gamma_j}\phi\left(\frac{\gamma_j}{2\pi}\log c_\pi\right),$$

where c_π is the analytic conductor of $L(s,\pi)$. It measures the density of zeros of $L(s,\pi)$ which are within $O(\frac{1}{\log c_\pi})$ of the central point $s=\frac12$.

Let $\mathfrak{F}(X)$ be the set of L-functions in $\mathfrak{F}$ such that $X<c_\pi<2X$. The one-level density conjecture says that

$$\lim_{X\to\infty}\frac{1}{|\mathfrak{F}(X)|}\sum_{\pi\in\mathfrak{F}(X)} D(\pi,\phi)=\int_{-\infty}^\infty \phi(x)W(G(\mathfrak{F}))\,dx,$$

where $W(G(\mathfrak{F}))$ is the one-level density function described below.

There are five possible symmetry types of families of L-functions: U, SO(even), SO(odd), O, and Sp. The corresponding density functions $W(G)$ are determined by Katz-Sarnak. They are

$$W(\mathrm{U})(x) = 1, \quad W(\mathrm{SO(even)})(x) = 1 + \frac{\sin 2\pi x}{2\pi x},$$
$$W(\mathrm{SO(odd)})(x) = 1 - \frac{\sin 2\pi x}{2\pi x} + \delta_0(x),$$
$$W(\mathrm{O})(x) = 1 + \frac{1}{2}\delta_0(x), \quad W(\mathrm{Sp})(x) = 1 - \frac{\sin 2\pi x}{2\pi x}.$$

By Plancherel's formula (and because ϕ is even),

$$\int_{-\infty}^{\infty} \phi(x) W(G)(x) dx = \int_{-\infty}^{\infty} \hat{\phi}(x) \widehat{W}(G)(x) dx.$$

It is useful to record that

$$\widehat{W}(\mathrm{U})(x) = \delta_0(x), \quad \widehat{W}(\mathrm{SO(even)})(x) = \delta_0(x) + \frac{1}{2}\chi_{[-1,1]}(x);$$
$$\widehat{W}(\mathrm{SO(odd)})(x) = \delta_0(x) - \frac{1}{2}\chi_{[-1,1]}(x) + 1;$$
$$\widehat{W}(\mathrm{O})(x) = \delta_0(x) + \frac{1}{2}; \quad \widehat{W}(\mathrm{Sp})(x) = \delta_0(x) - \frac{1}{2}\chi_{[-1,1]}(x).$$

There are many results which support the one-level density conjecture. Here an important question is to find natural families of L-functions. Iwaniec, Luo and Sarnak studied families of L-functions of newforms of even weight k and level N. Here one may fix the weight k and vary N or vice versa.

Consider the families of Artin L-functions. For a number field K of degree n, let $\widehat{K}$ be its Galois closure over $\mathbb{Q}$ so that $Gal(\widehat{K}/\mathbb{Q}) \simeq S_n$. We attach the Artin L-function $L(s,\rho,K) = \frac{\zeta_K(s)}{\zeta(s)}$, where ρ is a $n-1$-dimensional representation of S_n. Let

$$L(X) = \{L(s,\rho) : X < |d_K| < 2X, Gal(\widehat{K}/\mathbb{Q}) \simeq S_n\}.$$

Theorem 3.3 (Cho-K). *Let $n \leq 5$. Then $L(X)$ has Sp symmetry type, i.e.,*

$$\lim_{X\to\infty} \frac{1}{|L(X)|} \sum_{\pi \in L(X)} D(\pi,\phi) = \hat{\phi}(0) - \frac{1}{2}\phi(0) = \int_{-\infty}^{\infty} \phi(x)(1 - \frac{\sin 2\pi x}{2\pi x})\, dx,$$

where supp $\hat{\phi}$ is very small.

[In order to prove Katz-Sarnak conjecture, we need to show that it is true for arbitrary ϕ.]

3.3. **Transcendence of values of L-functions.** For a positive integer k and a Dirichlet character χ mod N such that $\chi(-1) = (-1)^k$, let $G_k(N, \chi)$ denote the space of all holomorphic modular forms $f(z)$ satisfying

$$f(\gamma(z)) = \chi(d)(cz + d)^k f(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

The subspace of $G_k(N, \chi)$ consisting of all cusp forms is denoted by $S_k(N, \chi)$. Further, we put

$$G_k(N) = \bigcup_\chi G_k(N, \chi), \quad S_k(N) = \bigcup_\chi S_k(N, \chi),$$

where χ runs over all characters mod N. These are the spaces of holomorphic modular forms and cusp forms of weight k with respect to the group

$$\Gamma_1(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 (\text{mod } N)\}.$$

Every element f of $G_k(N)$ has a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n q^n, \quad q = e^{2\pi i z}.$$

Put $D(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$. For an arbitrary Dirichlet character ψ, we put

$$D(s, f, \psi) = \sum_{n=1}^{\infty} \psi(n) a_n n^{-s}.$$

For a Dirichlet character ψ, let ψ_0 be the primitive character associated with ψ and c its conductor. Let

$$g(\psi) = g(\psi_0) = \sum_{n=1}^{c} \psi_0(n) e^{2\pi i n/c}$$

be the Gauss sum. For every positive integer $m < k$, we put

$$A(m, f, \psi) = (2\pi i)^{-m} g(\psi)^{-1} D(m, f, \psi).$$

With another Dirichlet character ψ' and another positive integer $m' < k$, we set

$$B(m, m'; f; \psi, \psi') = A(m, f, \psi)/A(m', f, \psi'),$$

assuming that $D(m', f, \psi') \neq 0$.

If $f(z)=\sum_{n=1}^{\infty} a_n q^n$ is primitive, then for every automorphism $\sigma \in Aut(\mathbb{C})$, we can define a primitive cusp form f^σ by

$$f^\sigma(z)=\sum_{n=1}^{\infty} a_n^\sigma q^n.$$

For f a primitive form, let $K_f=\mathbb{Q}(a_n, n=1,2,...)$ be the Hecke field. Let $K_\psi=\mathbb{Q}(\psi(n))$.

Theorem 3.4 (Shimura). *Let ψ,ψ' be primitive Dirichlet characters, and f a primitive cusp form belonging to $S_k(N,\chi)$. Let m,m' be positive integers less than k such that $(\psi\psi')(-1)=(-1)^{m-m'}$, and $D(m',f,\psi')\neq 0$. Then*

(1) If $k>2$, $B(m,m';f,\psi,\psi')\in K_f K_\psi K_{\psi'}$.
(2) For every $\sigma\in Aut(\mathbb{C})$, $D(m',f^\sigma,\psi'^\sigma)\neq 0$, and $B(m,m';f,\psi,\psi')^\sigma = B(m,m';f^\sigma,\psi^\sigma,\psi'^\sigma)$, where $\psi^\sigma(n)=\psi(n)^\sigma$.
(3) If $k=2$, the same assertions hold provided that f satisfies the following: for a given integer t, there is a primitive character ξ such that $D(1,f,\xi)\neq 0$ and $\xi(-1)=(-1)^t$. [This condition is always satisfied.]

An integer m is called critical for a motivic L-function $L(s,M)$ if both $L_\infty(s,M)$ and $L_\infty(1-s,M^\vee)$ are holomorphic at $s=m$. For example, in our case, we want $\Gamma(s)$ and $\Gamma(k-s)$ to be holomorphic. So the critical points are $0<m<k$.

Corollary 3.5. *As a special case, let f be a primitive cusp form belonging to $S_k(N,\chi)$. Let m,m' be positive integers less than k. Then $\pi^{-(m-m')}D(m,f)/D(m',f)\in K_f$, algebraic. When $k=2$, let f be a primitive cusp form belonging to $S_2(N,\chi)$. Let ψ,ψ' be primitive Dirichlet characters such that $D(1,f,\psi')\neq 0$. Then $D(1,f,\psi)/D(1,f,\psi')$ is algebraic.*

In particular, $m,m+1$ are critical, $\pi^{-1}D(m+1,f)/D(m,f)$ is algebraic. If m is a critical point, Shimura showed $D(m,f)\sim(2\pi i)^m\omega_\pm(f)$, where $(-1)^m=\pm$, and $<f,f>=\omega_+(f)\omega_-(f)$. Here $A\sim B$ means $A/B\in K_f$. We have the following conjecture due to Deligne:

Conjecture 3.6 (Deligne). *Suppose m is a critical point.*

(1)

$$L(m,Sym^{2l+1}f)\sim(2\pi i)^{m(l+1)}\omega_\pm(f)^{\frac{(l+1)(l+2)}{2}}\omega_\mp(f)^{\frac{l(l+1)}{2}}\delta(\chi)^{\frac{l(l+1)}{2}},$$

where $\delta(\chi)=(2\pi i)^{1-k}\sum_{u=0}^{c-1}\chi_0(u)e^{-\frac{2\pi i u}{c}}$, and χ_0 is the primitive character associated to χ with conductor c.

(2)

$$L(m, Sym^{2l}f) \sim \begin{cases} (2\pi i)^{m(l+1)}(\omega_+(f)\omega_-(f))^{\frac{l(l+1)}{2}}\delta(\chi)^{\frac{l(l+1)}{2}}, & \textit{if } m \textit{ even} \\ (2\pi i)^{ml}(\omega_+(f)\omega_-(f))^{\frac{l(l+1)}{2}}\delta(\chi)^{\frac{l(l-1)}{2}}, & \textit{if } m \textit{ odd} \end{cases}.$$

For $Sym^2 f$, it is due to Sturm:

$$L(m, Sym^2 f) \sim \begin{cases} (2\pi i)^{2m}(\omega_+(f)\omega_-(f))\delta(\chi), & \text{if } m \text{ even} \\ (2\pi i)^{m}(\omega_+(f)\omega_-(f)), & \text{if } m \text{ odd} \end{cases}.$$

For $Sym^3 f$, it is due to Garrett and Harris:

$$L(m, Sym^3 f) \sim (2\pi i)^{2m}\omega_\pm(f)^3\omega_\mp(f)\delta(\chi), \quad (-1)^m = \pm.$$

For $Sym^4 f$, it takes the form:

$$L(m, Sym^4 f) \sim \begin{cases} (2\pi i)^{3m}(\omega_+(f)\omega_-(f))^3\delta(\chi)^3, & \text{if } m \text{ even} \\ (2\pi i)^{2m}(\omega_+(f)\omega_-(f))^3\delta(\chi), & \text{if } m \text{ odd} \end{cases}.$$

For $f, h \in G_k(N)$, one of them being a cusp form, we can define the Petersson inner product $\langle f, h\rangle$ by

$$\langle f, h\rangle = m(\mathcal{F})^{-1}\int_{\mathcal{F}} \overline{f(z)}h(z)y^{k-2}\,dxdy, \quad z = x + iy,$$

where $\mathcal{F}$ is a fundamental domain for $\Gamma_1(N)$, and $m(\mathcal{F})$ is the measure of $\mathcal{F}$ with respect to $\frac{dxdy}{y^2}$. In fact, $m(\mathcal{F}) = \frac{\pi}{3}[SL_2(\mathbb{Z}) : \Gamma_1(N)\{\pm\}] = \frac{\pi}{3}N^2\prod_{p|N}(1 - \frac{1}{p^2})$.

Theorem 3.7. *Let f be a primitive element of $S_k(N, \chi)$, and ψ, ψ' be two primitive Dirichlet characters and m, m' be positive integers such that $(\psi\psi')(-1) = (-1)^{m-m'-1}$ and $0 < m, m' < k$. Put*

$$C(m, m'; f, \psi, \psi') = \frac{A(m, f, \psi)A(m', f, \psi')}{i^{1-k}\pi g(\chi)\langle f, f\rangle}.$$

Then $C(m, m'; f, \psi, \psi') \in K_f K\psi K_{\psi'}$. Moreover, for any $\sigma \in Aut(\mathbb{C})$,

$$C(m, m'; f; \psi, \psi')^\sigma = C(m, m'; f; \psi^\sigma, (\psi')^\sigma).$$

Let $f \in S_k(\Gamma)$, $\Gamma = SL_2(\mathbb{Z})$. Let I_f be the Ikeda lift in $S_{k+n}(Sp_{4n}(\mathbb{Z}))$ (rank $2n$ and $k + n$ even).

Then Choie-Kohnen, Furusawa and Kawamura-Katsurada showed that

$$\frac{< I_f, I_f >}{< f, f >^n} \in K_f.$$

For $f \in S_{2k-8}(\Gamma)$, we can construct the Ikeda type lift on the exceptional group of type $E_{7,3}$ of weight $2k$. Then

Theorem 3.8 (Katsurada-K-Yamauchi).

$$\frac{< F_f, F_f >}{< f, f >^3} \in K_f.$$

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