Arthur's classification for non-quasi-split odd special orthogonal groups

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Introduction

◇ In this talk, a representation means a representation over $\mathbb{C}$.

- Automorphic representations have been important objects in number theory.
- It is conjectured that one can classify them in terms of representations of the Galois group
  $\cdots$ "Global Langlands conjecture":

  Automorphic representations $\longleftrightarrow$ Representations of the Galois group

Introduction

- F: a number field, $\mathbb{A}_F$: the ring of adeles
- G: a connected reductive group over F
- Z: the center of G
- $L^2(G) := L^2(G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F))$
  - A representation of $G(\mathbb{A}_F)$ defined by $[g \cdot f](x) = f(xg)$
- $L^2(G) = L^2_{\mathrm{cont}}(G) \oplus L^2_{\mathrm{disc}}(G)$

A representation π of $G(\mathbb{A}_F)$ is called a <u>discrete automorphic representation</u> if it is a subrepresentation of $L^2_{\mathrm{disc}}(G)$.

$\rightsquigarrow$"Global Langlands conjecture" can be stated as the following style:

$$L^2_{\mathrm{disc}}(G) = \bigoplus_{R:\text{representations of } \mathrm{Gal}(\overline{F}/F)} \pi_R$$

◀ $\pi_R \subset L^2_{\mathrm{disc}}(G)$, irreducible.

◦ For simplicity, we do not consider central characters.

Introduction

- Arthur proved it for $G =$ quasi-split $\mathrm{Sp}(2n)$ or $\mathrm{SO}(N)$ under the assumption that it holds for GL_k.
- He classified discrete automorphic representations of $G(\mathbb{A}_F)$ in terms of discrete automorphic representations of GL_k $\cdots$ Arthur's classification:

$$L^2_{\mathrm{disc}}(G) = \bigoplus_{\substack{r:\text{parameters related to discrete} \\ \text{automorphic representations of } \mathrm{GL}_k(\mathbb{A}_F)}} \pi_r.$$

Also known for $G =$

- quasi-split $U(N)$ $\cdots$ Mok
- non-quasi-split $U(N)$ $\cdots$ Kaletha–Mínguez–Shin–White ("generic part")

Main result: Arthur's classification for non-quasi-split $\mathrm{SO}(2n+1)$

Introduction

- Arthur treated quasi-split $\mathrm{Sp}(2n)$ and $\mathrm{SO}(N)$ $\cdots$ Original work
- Mok treated quasi-split $U(N)$ $\cdots$ Analogue of Arthur's work
- KMSW treated inner forms of $U(N)$ $\cdots$ Reduction to Mok's result (partially analogous to Arthur's work)
- Today, we treat inner forms of $\mathrm{SO}(2n+1)$ $\cdots$ Analogue of KMSW, i.e., reduction to Arthur's result

○ Why $\mathrm{SO}(2n+1)$? (Not $\mathrm{Sp}(2n)$, $\mathrm{SO}(2n)$)

▷ Application for Arthur's classification for metaplectic groups (Gan–Ichino)

Main theorem

Notaiton

- F: a number field

For any place v of F,

- F_v: the localization
- q_v: the residual order of F_v (if $v < \infty$)
- W_{F_v}: the Weil group of F_v
- $L_{F_v} := \begin{cases} W_{F_v}, & v\text{: archimedean,} \\ W_{F_v} \times \mathrm{SL}_2(\mathbb{C}), & v\text{: non-archimedean} \end{cases}$
- $\mathbb{A}_F$: the ring of adeles of F

Near equivalence

- SO_{2n+1}: the split special orthogonal group of degree $2n+1$, over F
- G: an inner form of SO_{2n+1}

Definition of near equivalence

Two irreducible representations $\pi = \otimes_v \pi_v$ and $\pi' = \otimes_v \pi'_v$ of $G(\mathbb{A}_F) = \prod'_v G(F_v)$ are nearly equivalent if $\pi_v \simeq \pi'_v$ for almost all v.

$$\rightsquigarrow \quad L^2_{\mathrm{disc}}(G) = \bigoplus_{C:\text{near equivalence classes}} C$$

Divide the main theorem into two parts

- 1st: parametrization of $\{C\}$;
- 2nd: parametrization of irreducible components in each C.

A-parameter

Definition of A-parameter for SO_{2n+1}

An <u>A-parameter for G</u> is a formal unordered sum

$$\psi = \ell_1(\mu_1 \boxtimes S_{d_1}) \oplus \cdots \oplus \ell_r(\mu_r \boxtimes S_{d_r}),$$

where $\ell_i \geq 1$ is an integer, and

- μ_i: an irreducible cuspidal automorphic representation of $\mathrm{GL}_{m_i}(\mathbb{A}_F)$;
- S_{d_i}: the unique irreducible d_i-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$;
- $i \neq j \Rightarrow (\mu_i, d_i) \neq (\mu_j, d_j)$;
- $\sum_i \ell_i m_i d_i = 2n$;
- ℓ_i is even unless one of μ_i and S_{d_i} is symplectic and the other is orthogonal.

- $\Psi(G) := \{A\text{-parameter } \psi \text{ for } G\}$

The 1st main theorem

Theorem 1

For an A-parameter ψ for G, let $L^2_\psi(G)$ be the near equivalence class defined below. Then

$$L^2_{\mathrm{disc}}(G) = \bigoplus_{\psi \in \Psi(G)} L^2_\psi(G).$$

Definition of $L^2_\psi(G)$

- $\psi = \ell_1(\mu_1 \boxtimes S_{d_1}) \oplus \cdots \oplus \ell_r(\mu_r \boxtimes S_{d_r}) = \bigoplus_i \mu_i \boxtimes S_{d_i}$

$\rightsquigarrow$ $\psi_v := \bigoplus_i \mu_{i,v} \boxtimes S_{d_i}$, for any place v of F

- $\mu_{i,v}$: an irreducible representation of $\mathrm{GL}_{m_i}(F_v)$
  $\cdots$ unramified for almost all v
- $\{\alpha^{(j)}_{i,v}\}_j$: the Satake parameter of $\mu_{i,v}$

$\rightsquigarrow$ $\pi \subset L^2_\psi(G) \overset{def}{\Leftrightarrow}$ the Satake parameter of π_v is $\{\alpha^{(j)}_{i,v} q_v^{-\frac{d_i-1}{2}+k}\}_{i,j,0 \leq k \leq d_i-1}$, for almost all v.

Another equivalent definition of $L^2_\psi(G)$

- $\psi = \ell_1(\mu_1 \boxtimes S_{d_1}) \oplus \cdots \oplus \ell_r(\mu_r \boxtimes S_{d_r})$: an A-parameter for G

$\rightsquigarrow$ $\psi_v = \ell_1(\mu_{1,v} \boxtimes S_{d_1}) \oplus \cdots \oplus \ell_r(\mu_{r,v} \boxtimes S_{d_r})$

- ◀ $\mu_{i,v}$: a representation of $\mathrm{GL}_{m_i}(F_v)$
  $\rightsquigarrow$ regarded as an L-parameter via LLC for $\mathrm{GL}_{m_i}(F_v)$
- ▶ $\mu_{i,v} \colon L_{F_v} \to \mathrm{GL}_{m_i}(\mathbb{C})$

▶ $\psi_v \colon L_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2n}(\mathbb{C})$

$\rightsquigarrow$ $\phi_{\psi_v}(w) := \psi_v(w, \begin{pmatrix} |w|^{\frac{1}{2}} & \\ & |w|^{-\frac{1}{2}} \end{pmatrix})$

$\rightsquigarrow$ $\pi \subset L^2_\psi(G) \overset{def}{\Leftrightarrow}$ the Satake parameter of π_v equals to that of ϕ_{ψ_v} for almost all v

Local packets

- ψ: an A-parameter for $G \rightsquigarrow \psi_v \colon L_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2n}(\mathbb{C})$
- $\mathcal{S}_{\psi_v} := \pi_0(C_{\psi_v})$, $C_{\psi_v} := \mathrm{Cent}(\mathrm{Im}\,\psi_v, \mathrm{Sp}_{2n}(\mathbb{C}))$
- $\hat{\mathcal{S}}_{\psi_v}$: the group of characters of $\mathcal{S}_{\psi_v}$

We can define the local packet

- $\Pi_{\psi_v}(G_v)$: a finite multiset of irreducible unitary representations of $G_v(F_v)$
  - with a map $\Pi_{\psi_v}(G_v) \ni \pi_v \mapsto \langle -, \pi_v \rangle \in \hat{\mathcal{S}}_{\psi_v}$

◀ $\Pi_{\psi_v}(G_v)$ and $\langle -, \pi_v \rangle$ are characterized by "Endoscopic character celation" (next page) and induction of representations.

◁ Construction of them is not so explicit.

◁ Whittaker datum for SO_{2n+1} is unique.

Endoscopic character relation

Assume G_v: non-quasi-split. (G_v: quasi-split $\Rightarrow$ Arthur's work) Choose

- $s \in C_{\psi_v}$ with $Z_{\mathrm{Sp}_{2n}(\mathbb{C})}(s) \simeq \mathrm{Sp}_{2a}(\mathbb{C}) \times \mathrm{Sp}_{2b}(\mathbb{C})$, $a + b = n$.

$\Rightarrow$ $\exists \psi_v^\varepsilon \colon L_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2\varepsilon}(\mathbb{C})$, $(\varepsilon = a, b)$ s.t. $\eta \circ (\psi_v^a \oplus \psi_v^b) = \psi_v$

- $\eta \colon \mathrm{Sp}_{2a}(\mathbb{C}) \times \mathrm{Sp}_{2b}(\mathbb{C}) \simeq Z_{\mathrm{Sp}_{2n}(\mathbb{C})}(s) \hookrightarrow \mathrm{Sp}_{2n}(\mathbb{C})$

Endoscopic character relation

Suppose $\psi_v(W_F)$ is bounded. Let $f \in C_0^\infty(G_v)$ and $f^\varepsilon \in C_0^\infty(\mathrm{SO}_{2\varepsilon+1})$. If f and $f^a \times f^b$ have "matching orbital integrals", then

$$\prod_{\varepsilon = a,b} \left(\sum_{\pi^\varepsilon \in \Pi_{\psi_v^\varepsilon}(\mathrm{SO}_{2\varepsilon+1})} \langle s_{\psi_v}^{(\varepsilon)}, \pi^\varepsilon \rangle \, \mathrm{tr}(\pi^\varepsilon(f^\varepsilon)) \right) = e(G_v) \sum_{\pi \in \Pi_{\psi_v}(G_v)} \langle s s_{\psi_v}, \pi \rangle \, \mathrm{tr}(\pi(f)),$$

where $s_{\psi_v} := \psi_v(1, -\mathbf{1}) =: s_{\psi_v}^{(a)} \times s_{\psi_v}^{(b)}$, and $e(G_v)$ is the Kottwitz sign.

Global packets

- $\psi = \ell_1(\mu_1 \boxtimes S_{d_1}) \oplus \cdots \oplus \ell_r(\mu_r \boxtimes S_{d_r}) \in \Psi(G)$: an A-parameter for G
- $\mathcal{S}_\psi := (\mathbb{Z}/2\mathbb{Z})a_1 \oplus \cdots \oplus (\mathbb{Z}/2\mathbb{Z})a_r$, $\quad$ (a_i: formal basis)
    - a canonical map $\mathcal{S}_\psi \to \mathcal{S}_{\psi_v}$, for all v
- $\Delta\colon \mathcal{S}_\psi \to \prod_v \mathcal{S}_{\psi_v}$: the diagonal map

We can define the global packet

- $\Pi_\psi(G) := \{\pi = \otimes_v \pi_v \mid \pi_v \in \Pi_{\psi_v}(G_v), \langle -, \pi_v \rangle = 1 \text{ for almost all } v\}$
    - with a map $\Pi_\psi(G) \ni \pi \mapsto \langle -, \pi \rangle := \prod_v \langle -, \pi_v \rangle \in (\widehat{\prod_v \mathcal{S}_{\psi_v}})$

The 2nd main theorem

- $\psi = \ell_1(\mu_1 \boxtimes S_{d_1}) \oplus \cdots \oplus \ell_r(\mu_r \boxtimes S_{d_r}) \in \Psi(G)$

We shall call an A-parameter ψ discrete if $\ell_i = 1$ for every $i = 1, \ldots, r$.

- $\Psi_2(G) := \{\text{discrete } A\text{-parameter for } G\} \subset \Psi(G)$
  - $\psi \in \Psi_2(G) \Leftrightarrow$ every $\mu_i \boxtimes S_{d_1}$ is symplectic and multiplicity free.

Theorem 2

If $d_i = 1$ for all $i = 1, \ldots, r$, then

$$L^2_\psi(G) = \begin{cases} 0, & \psi \notin \Psi_2(G), \\ \displaystyle\bigoplus_{\substack{\pi \in \Pi_\psi(G) \\ \langle -,\pi\rangle \circ \Delta = 1}} \pi, & \psi \in \Psi_2(G). \end{cases}$$

- If $d_i = 1$ for all $i = 1, \ldots, r$, the parameter ψ is said to be generic.

Remark on the non-generic part

Expectation

For every A-parameter $\psi \in \Psi(G)$, there is a character $\varepsilon_\psi \in \hat{\mathcal{S}}_\psi$ such that

$$L^2_\psi(G) = \begin{cases} 0, & \psi \notin \Psi_2(G), \\ \bigoplus\limits_{\substack{\pi \in \Pi_\psi(G) \\ \langle -, \pi \rangle \circ \Delta = \varepsilon_\psi}} \pi, & \psi \in \Psi_2(G), \end{cases}$$

and ε_ψ can be given explicitly.

- In the quasi-split cases (Arthur, Mok), the expectation above was also proved.
- A paper by KMS about it for unitary groups is now in preparation.

Outline of the proof

Main tools

- Induction on the size of the group ($\rightarrow$We assume all theorems holds for any proper Levi subgroup.)
- Trace formula and its stabilization
- Local intertwining relation

Proof of Theorem 1

- First, we have $L^2_{\mathrm{disc}}(G) = \bigoplus_{(t,c)} L^2_{t,c}(G)$
  - $t \in \mathbb{R}_{\geq 0}$, $c = (c_v)_v \in \prod_v (\mathbb{C}^\times)^n / W(G)$
  - $L^2_{t,c}(G) := \bigoplus \pi$, where π runs over s.t. the Hermitian norm of the imaginary part of the infinitesimal character of the archimedean component π_∞ (a representation of $G(\mathbb{C} \otimes_{\mathbb{Q}} F)$) is t, and the Satake parameter of π_v is c_v for almost all finite place v.

- One can associate $t(\psi)$ and $c(\psi)$ with $\psi \in \Psi(G)$ in the natural way.

$\rightsquigarrow$ Theorem 1: "$L^2_{\mathrm{disc}}(G) = \bigoplus_{\psi \in \Psi(G)} L^2_\psi(G)$" can be rephrased as

Theorem 1'

$L^2_{t,c}(G) = 0$ unless $(t,c) = (t(\psi), c(\psi))$ for some $\psi \in \Psi(G)$.

Proof of Theorem 1' -Trace Formula-

- R_{disc}: the representation of $G(\mathbb{A}_F)$ on $L^2_{\mathrm{disc}}(G)$
- $R_{\mathrm{disc}} = \bigoplus_{(t,c)} R_{t,c}$, parallel to $L^2_{\mathrm{disc}}(G) = \bigoplus_{(t,c)} L^2_{t,c}(G)$

For any (t, c), consider

- the discrete part of the trace formula:

$$I^G_{t,c}(f) := \sum_{\substack{M \subset G \\ \text{Levi}}} \sum_{w \in W^G_{M,\mathrm{reg}}} C_{M,w} \cdot \mathrm{tr}(M_{P,t,c}(w) \circ \mathcal{I}_{P,t,c}(f)), \ f \in C^\infty_0(G(\mathbb{A}_F))$$

  - P: a parabolic subgroup of G with the Levi subgroup M
  - $\mathcal{I}_{P,t,c}$: the "(t, c)-part" of the representation induced from $L^2_{\mathrm{disc}}(M)$.
  - $M_{P,t,c}(w)$: an intertwining operator
  - $C_{M,w}$: a constant

- ◦ $\mathrm{tr}\, R_{t,c}(f)$ is the summand for $M = G$

Proof of Theorem 1' -Stable Multiplicity Formula-

- The stabilization of $I_{t,c}^{G}(f)$:

$$I_{t,c}^{G}(f) = \sum_{G':\ \text{elliptic endoscopic group of } G/\simeq} \iota(G, G') \cdot S_{t,c}^{G'}(f')$$

- ◀ G' runs over $\{\mathrm{SO}_{2a+1} \times \mathrm{SO}_{2b+1} \mid a + b = n\}$
- ◀ $\iota(G, G')$: a constant
- ◀ $f' \in C_0^{\infty}(G')$: the "transfer" of f
- ◀ $S_{t,c}^{G'}(f')$: stable linear form

Theorem (Stable multiplicity formula) (Arthur)

If there is an A-parameter $\psi \in \Psi(G)$ such that $(t, c) = (t(\psi), c(\psi))$, then

$$S_{t,c}^{G'}(f) = |S_{\psi}(G')|^{-1} \varepsilon_{\psi}^{G'}(s_{\psi}) \sigma(\overline{S}_{\psi}^{\circ}(G')) f'(\psi).$$

Otherwise, $S_{t,c}^{G'}(f) = 0$.

Proof of Theorem 1' -induction-

Fix (t, c).
Suppose that there is no $\psi \in \Psi(G)$ such that $(t, c) = (t(\psi), c(\psi))$.
$\Rightarrow$ $I^G_{t,c}(f) = 0$ ($\because$ Stable multiplicity formula)

- Induction hypothesis: All theorems holds for groups of smaller size.

$\rightsquigarrow$ All theorems holds for every proper Levi subgroup

- Recall:

$$I^G_{t,c}(f) := \sum_{\substack{M \subset G \\ \text{Levi}}} \sum_{w \in W(G,M)_{\mathrm{reg}}} C_{M,w} \cdot \mathrm{tr}(M_{P,t,c}(w) \circ \mathcal{I}_{P,t,c}(f))$$

$\rightsquigarrow$ Every term for $M \subsetneq G$ vanishes.
$\rightsquigarrow$ $I^G_{t,c}(f) = \mathrm{tr}\, R_{t,c}(f)$

$\Rightarrow$ $\mathrm{tr}\, R_{t,c}(f) = 0$, which means $L^2_{t,c}(G) = 0$

Proof of Theorem 2 -the difference ${}^0r_\psi$-

Recall Theorem 2: For a generic A-parameter $\psi \in \Psi(G)$, we have

$$L^2_\psi(G) = \begin{cases} 0, & \psi \notin \Psi_2(G), \\ \displaystyle\bigoplus_{\substack{\pi \in \Pi_\psi(G) \\ \langle -, \pi \rangle \circ \Delta = \varepsilon_\psi}} \pi, & \psi \in \Psi_2(G). \end{cases}$$

We shall show that

$${}^0r_\psi(f) := \mathrm{tr}\, R_\psi(f) - \mathrm{tr}(\mathrm{RHS})(f)$$

is identically zero.

Proof of Theorem 2 -the Standard Model-

Arthur studied $I_\psi^G(f) := I_{t(\psi),c(\psi)}^G(f)$ more deeply.

- Spectral terms:

$$I_\psi^G(f) := \sum_{\substack{M\subset G\\ \text{Levi}}} \sum_{w\in W(G,M)_{\mathrm{reg}}} C_{M,w} \cdot \mathrm{tr}(M_{P,\psi}(w)\circ \mathcal{I}_{P,\psi}(f))$$

$$\cdots = {}^0 r_\psi(f) + \sum_{s\in\mathcal{S}_\psi} \kappa_{\psi,s} f_G(\psi,s)$$

  - $\kappa_{\psi,s}$: a constant
  - $f_G(\psi,s)$: a linear form related to an intertwining operator

- Endoscopic terms:

$$I_\psi^G(f) = \sum_{G'\text{: ell. endos. of } G/\simeq} \iota(G,G') S_\psi^{G'}(f')$$

$$\cdots = \sum_{s\in\mathcal{S}_\psi} \kappa_{\psi,s} f_G'(\psi, s^{-1}s_\psi)$$

  - $f_G'(\psi, s^{-1}s_\psi)$: a stable linear form on an endoscopic group

Proof of Theorem 2 -Intertwining Relations-

$\rightsquigarrow$ The proof is reduced to

Global intertwining relation

For any $s \in \mathcal{S}_\psi$,

$$f_G(\psi, s) = f'_G(\psi, s^{-1}s_\psi).$$

- For $f = \bigotimes_v f_v \in C_0^\infty(G(\mathbb{A}_F))$, one can decompose
  - $f_G(\psi, s) = \prod_v f_{v,G_v}(\psi_v, s_v)$;
  - $f'_G(\psi, s^{-1}s_\psi) = \prod_v f'_{v,G_v}(\psi_v, s_v^{-1}s_{\psi,v})$.

$\rightsquigarrow$ The proof is reduced to

Local intertwining relation

For any place v,

$$f_{v,G_v}(\psi_v, s_v) = f'_{v,G_v}(\psi_v, s_v^{-1}s_{\psi,v}).$$

Proof of Theorem 2 -Proof of Local Intertwining Relation-

- F: a local field
- G: a non-quasi-split inner form of SO_{2n+1}
- ψ: a "local A-parameter" for G

Local intertwining relation

For some special $s \in C_\psi = \mathrm{Cent}(\mathrm{Im}\,\psi, \mathrm{Sp}_{2n}(\mathbb{C}))$,

$$f_G(\psi, s) = f'_G(\psi, s^{-1}s_\psi).$$

Key idea

- Reduction of the proof to the case of some special kind of parameters ψ
- Globalization (a problem in non-generic case) $\cdots$ Next page

Proof of Theorem 2 -Proof of Local Intertwining Relation-

- $(\dot{F}, u, \dot{G}, \dot{\psi})$: a globalization of (F, G, ψ) s.t.
    - $\dot{F}_u = F$, $\dot{G}_u = G$, $\dot{\psi}_u = \psi$;
    - $\exists w$ s.t. $\dot{F}_w = \mathbb{R}$, $\dot{G}_v$ is split for $v \neq u, w$;
    - "Quasi"-global intertwining relation holds for $(\dot{F}, \dot{G}, \dot{\psi})$;
    - etc.

$\rightsquigarrow$ The proof is reduced to some special concrete cases over $\mathbb{R}$

- Direct calculations complete the proof.

Thank you

$$L^2_{\mathrm{disc}}(G) = \bigoplus_{\psi \in \Psi(G)} L^2_\psi(G).$$

$$L^2_\psi(G) = \begin{cases} 0, & \psi \notin \Psi_2(G), \\ \displaystyle\bigoplus_{\substack{\pi \in \Pi_\psi(G) \\ \langle -, \pi \rangle \circ \Delta = \varepsilon_\psi}} \pi, & \psi \in \Psi_2(G), \end{cases}$$