On the essential logical structure of inter-universal Teichmüller theory

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§1. Isogenies of elliptic curves and global multiplicative subspaces/canonical generators

- A special case of Faltings' isogeny invariance of the ht for elliptic curves

Key assumption: ∃ global multiplicative subspace (GMS)

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Recall: For l a prime number, the module of l-torsion points assoc. to a Tate curve $E \stackrel{\text{def}}{=} \mathbb{G}_m/q^{\mathbb{Z}}$ — over, say, $\mathbb{C}$ or a p-adic local field — fits into a natural exact sequence:

$$1 \longrightarrow \boldsymbol{\mu}_l \longrightarrow E[l] \longrightarrow \mathbb{Z}/l\mathbb{Z} \longrightarrow 1.$$

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$$1 \longrightarrow \boldsymbol{\mu}_l \longrightarrow E[l] \longrightarrow \mathbb{Z}/l\mathbb{Z} \longrightarrow 1.$$

Thus, one has canonical objects as follows:

a "multiplicative subspace" $\boldsymbol{\mu}_l \subseteq E[l]$ and "generators" $\pm 1 \in \mathbb{Z}/l\mathbb{Z}$.

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$\Longrightarrow$ In general, $E[l]$ does not admit

a "global multiplicative subspace" and "generators"

that coincide w/ the above canonical "multiplicative subspace" and "generators" at all finite places where E has bad multiplicative red'n!

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Then, if we denote by

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If we write $\log(q_E)$, $\log(q_E^*)$ for the arithmetic degrees $\in \mathbb{R}$ det'd by these q-parameters, then the above rel'n takes on the following form:

$$l \cdot \log(q_E) = \log(q_{E^*}) \in \mathbb{R}.$$

On the other hand, if we consider the respective heights of the elliptic curves by $\mathrm{ht}_E, \mathrm{ht}_{E^*} \in \mathbb{R}$ — i.e, roughly speaking, arithmetic degrees of arithmetic line bundles on F

$$\omega_E^{\otimes 2}, \quad \omega_{E^*}^{\otimes 2},$$

assoc. to the sheaves of square differentials — then we may conclude — cf. the disc. mod. form, regarded as a section of the ample line bundle "$\omega_{\overline{\mathcal{M}}_{\mathrm{ell}}}^{\otimes 12}$" on the compactified moduli stack $\overline{\mathcal{M}}_{\mathrm{ell}}$ of elliptic curves! — that

$$\mathrm{ht}_{(-)} \approx \tfrac{1}{6} \cdot \log(q_{(-)})$$

(where "$\approx$" means "up to a discrepancy bounded by a constant").

Moreover, by the famous computation concerning differentials due to Faltings (1983), one knows that:

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$$l \cdot \mathrm{ht}_E \lesssim \mathrm{ht}_E + \log(l); \quad \text{i.e.,} \quad \mathrm{ht}_E \lesssim \frac{1}{l-1} \cdot \log(l) \lesssim \text{constant}$$

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— that is to say, that the height ht_E can be bounded from above, and hence (under suitable hypotheses) that there are only finitely many isom. classes of elliptic curves E that admit a "global multiplicative subspace".

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... $\rightsquigarrow$ need not only GMS, but also

... global canonical generators (GCG) (cf. §5)!

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- One way to summarize IUT:

to generalize the above approach to bounding heights
via theta functions + anabelian geometry
to the case of arbitrary elliptic curves
by somehow "simulating" GMS + GCG!

§2. Gluings via Teichmüller dilations, inter-universality, and logical $\wedge/\vee$

- Naive approach to generalizing Frobenius aspect "$q^l \approx q$" of §1

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for $N \geq 2$ an integer, p a prime number, glue via "$*$" (cf. [Alien], §3.11, (iv); [EssLgc], Example 3.1.1; [EssLgc], §3.4):

$$^{\dagger}\mathbb{Z} \ni {}^{\dagger}p^N \leftarrow: * :\rightarrow {}^{\ddagger}p \in {}^{\ddagger}\mathbb{Z} \quad \text{. . . so} \quad (* \mapsto {}^{\dagger}p^N \in {}^{\dagger}\mathbb{Z}) \ \wedge \ (* \mapsto {}^{\ddagger}p \in {}^{\ddagger}\mathbb{Z})$$

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... not compatible with ring structures!!

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. . . but compatible with multiplicative structures, actions of Galois groups as abstract groups!!

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... ultimately, we want to delete labels (cf. §1!), but doing so naively yields — if one is to avoid giving rise to a contradiction "$p^N = p$"! — a meaningless OR "$\vee$" indeterminacy!!

$$ (* \mapsto p^N \in \mathbb{Z}) \ \vee \ (* \mapsto p \in \mathbb{Z}) \qquad \Longleftrightarrow \qquad * \mapsto ?? \in \{p, p^N\} \subseteq \mathbb{Z} $$

(cf. "contradiction" asserted by "redundant copies school (RCS)"!)

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... in IUT, we would like to delete the labels in a somewhat more "constructive" (!) way!

$l \geq 5$: a prime number; $l^{*} \stackrel{\mathrm{def}}{=} \frac{l-1}{2}$;

$E\ (= E_F)$: an elliptic curve over a number field F s.t. ...;

$E[l] \subseteq E$: the subgroup scheme of l-torsion points; $K \stackrel{\mathrm{def}}{=} F(E[l])$;

j_E: the j-invariant of E, so $F_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} \mathbb{Q}(j_E) \subseteq F$;

$\underline{\mathbb{V}} \subseteq \mathbb{V}(K)$: a collection of valuations of K s.t. ...;

$q_{\underline{v}}$: the local q-parameter of E at bad (nonarch.) $\underline{v} \in \underline{\mathbb{V}}$;

$G_{\underline{v}}$: the (local) abs. Gal grp of $K_{\underline{v}}$ regarded "inter-universally" as an abstract top. group, i.e., not as a ("Galois"!) group of field automorphisms (cf. incompatibility with ring structure!);

$\mathcal{O}_{\underline{\tilde{v}}}^{\times}$: units of the ring of integers $\mathcal{O}_{\underline{\tilde{v}}}$ of an algebraic closure $K_{\underline{\tilde{v}}}$ of the completion $K_{\underline{v}}$ of K at $\underline{v}$;

$\mathcal{O}_{\underline{\tilde{v}}}^{\times \boldsymbol{\mu}} \stackrel{\mathrm{def}}{=} \mathcal{O}_{\underline{\tilde{v}}}^{\times}/\text{tors} +$ "integral str." $\{\mathrm{Im}((\mathcal{O}_{\underline{\tilde{v}}}^{\times})^{H})\}_{\text{open } H \subseteq G_{\underline{v}}}$

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loc. unit gps.: $G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\underline{\tilde{v}}} \stackrel{\sim}{\to} G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\boldsymbol{\mu}}_{\underline{\tilde{v}}}$

loc. val. gps.: $\left(\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j=1,\ldots,l^*}\right)^{\mathbb{N}} \stackrel{\sim}{\to} \left(\underline{\underline{q}}_{\underline{v}} \stackrel{\mathrm{def}}{=} q_{\underline{v}}^{\frac{1}{2l}}\right)^{\mathbb{N}}$

glob. val. gps.: corresponding global realified Frobenioids (s.t. product formula holds!)

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the proj. line as a gluing of ring schemes along a multip. group scheme

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classical complex Teichmüller deformations of holomorphic structure

... cf. two combinatorial/arithmetic dimensions of a ring!!

... cf. assertions of the RCS!

cf. Complex Teichmüller theory (cf. [Pano], § 2; [Alien], § 3.3, (ii))

Rel. to a canonical coordin. $z = x + iy$ assoc'd to a square differential — on a Riemann surface, Teichmüller deformations given by

hol. str. $\curvearrowright$ $\mathbb{C} \ni \quad z \mapsto \zeta = \xi + i\eta = \lambda x + iy \quad \in \mathbb{C}$ $\curvearrowleft$ hol. str.

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Key points:

- non-hol. map, but common real analytic str. — i.e., "$\wedge$"!
- one hol. dim., but two underlying real dims., of which one is dilated/deformed, while the other is left fixed/undeformed!

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(cf. [Alien], §3.3, (ii), (vi), Fig. 3.6; [EssLgc], §3.3, (InfH); [EssLgc], §3.11, (ΘORInd), ($\mathfrak{log}$ORInd), (Di/NDi)):

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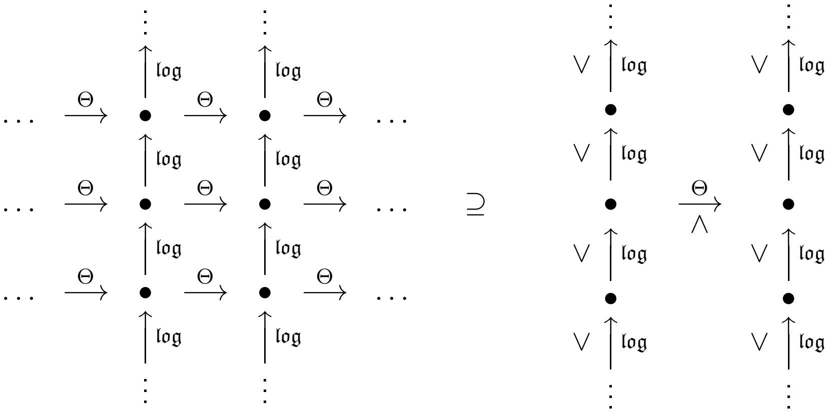
... but for Θ-link, this yields meaningless (ΘORInd)!!
$\Longrightarrow$ consider "saturation" ($\mathfrak{log}$ORInd) for $\mathfrak{log}$-link, i.e., by constructing invariants for $\mathfrak{log}$-link, where we recall that

$$\mathfrak{log} : \text{nondilated unit groups} \quad \rightleftarrows \quad \text{dilated value groups}$$

... i.e., for invariants, "nondilated $\Longleftrightarrow$ dilated" ... cf. proof of §1!!

The log-theta-lattice and the "infinite H" portion that is actually used:



$$\left(\text{i.e., } \textbf{not} \quad \underset{\vee}{\xrightarrow{\Theta}} \quad ! \right)$$

§3. Symmetries/nonsymmetries and coricities of the log-theta-lattice

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Θ-link (i.e., "$q^N \leftarrow: q$" for some $N \geq 2$) and $\mathfrak{log}$-link (i.e., "p-adic logarithm" for some p) satisfy the following (cf. [EssLgc], Example 3.2.2):

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(1) Θ-link, $\mathfrak{log}$-link are not compatible with the ring structures in their domains/codomains;

(2) Θ-link, $\mathfrak{log}$-link are not symmetric with respect to switching their domains/codomains;

(3) Θ-link ∘ $\mathfrak{log}$-link $\neq$ $\mathfrak{log}$-link ∘ Θ-link;

(4) Θ-link ∘ $\mathfrak{log}$-link $\neq$ Θ-link

- Frobenius-like objects: objs. whose dfn. depends, a priori, on the coord. "$(n, m) \in \mathbb{Z} \times \mathbb{Z}$" of the Hodge theater at which they are defined (e.g., rings, monoids, etc. that do not map isomorphically via Θ-link, $\mathfrak{log}$-link)

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• Étale-like objects: objs. that arise from arithmetic (étale) fund. groups regarded as abstract topological gps. . . . cf. inter-universality!

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$\Longrightarrow$ mono-anabelian absolute anabelian geometry may be applied (cf. ampleness of ω_E in §1!)

e.g.: inside each Hodge theater "•", at each $\underline{v}$, $\exists$ a copy of the arithmetic/tempered fundamental group

$$\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$$

of a certain fin. ét. cover. of $X_{\underline{v}} \stackrel{\text{def}}{=} E_{\underline{v}} \setminus \{\text{origin}\}$ (where $E_{\underline{v}} \stackrel{\text{def}}{=} E \times_F K_{\underline{v}}$)

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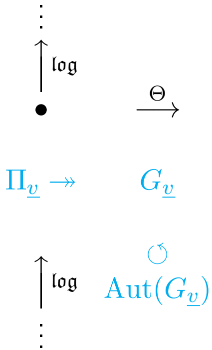
$$G_{\underline{v}} \xrightarrow{\sim} G_{\underline{v}}$$

— i.e., "(Ind1)" — between copies in dom./codom. of the Θ-link

(so abstract top. gps. $\Pi_{\underline{v}}$, $G_{\underline{v}}$ are coric for $\mathfrak{log}$-, Θ-links!) and symmetry properties:



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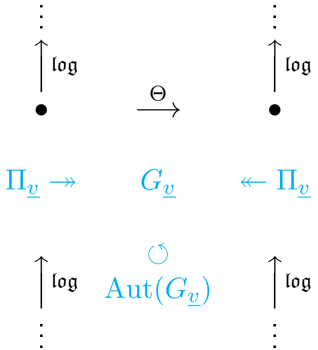

. . . symmetric w.r.t. dom./codom. of Θ-link!

In summary, w. r. t. the desired **symmetry** and **coricity** properties:

Frobenius-like	FALSE	FALSE
étale-like	TRUE	TRUE

§4. Frobenius-like vs. étale-like structures and Kummer-detachment indeterminacies

- Kummer theory yields isoms. between corresponding objects:

$$\text{Frobenius-like objects} \quad \stackrel{\sim}{\to} \quad \text{(mono-anabelian) étale-like objects}$$

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Frobenius-like objects that do not satisfy coricity/symmetry properties

to étale-like objects that do satisfy coricity/symmetry properties

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(c) for global field of moduli F_{mod}: Kummer theory via "κ-coric" algebraic rational functions (essentially, non-linear polynomials w.r.t. some "point at infinity") and Galois evaluation at points defined over number fields (cf. [Alien], Example 2.13.1; [Alien], §3.4, (ii))

- In general, "Kummer theory" proceeds by:

$$\begin{pmatrix} \text{extracting} \\ n\text{-th roots} \in M, \\ \text{for } n \in \mathbb{Z}_{>0}, \text{ of} \\ \text{some element} \\ f \in \text{a multipl.} \\ \text{monoid } M \end{pmatrix} \rightsquigarrow \begin{pmatrix} \text{Kummer class } \kappa_f \\ \in H^1\left(\begin{bmatrix} \text{some "Gal. group"} \\ \Pi \text{ that acts on } M \end{bmatrix}, \mu_n(M) \right) \end{pmatrix}$$

... where $\mu_n(M)$ denotes n-torsion — i.e., roots of unity! — of M;

$\rightsquigarrow$ "$\widehat{\mathbb{Z}}$ version" by taking $\varprojlim_{n}$

• Main Substantive Issue: eliminating potential $\widehat{\mathbb{Z}}^{\times}$-indeterminacy from the conventional cyclotomic rigidity isomorphism (CRI)

$$(\widehat{\mathbb{Z}} \cong) \quad \mu_{\widehat{\mathbb{Z}}}(M) \quad \stackrel{\sim}{\to} \quad \mu_{\widehat{\mathbb{Z}}}(\Pi) \quad (\cong \widehat{\mathbb{Z}})$$

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... note that this is a very substantive issue! indeed,

indeterminate $\widehat{\mathbb{Z}}^{\times}$-multiples/powers of divs., line bdls., rational/merom. fns., elts. of number fields/local fields

completely destroy any notion of positivity/inequalities
(recall that -1 lies in the closure of the natural numbers in $\widehat{\mathbb{Z}}$!)
for arithmetic degrees/heights;

moreover, inter-universality — i.e., the property of not being anchored to/rigidified by any particular ring/sch. theory" — means that the $\mathcal{O}_{\underline{\tilde{v}}}^{\times\boldsymbol{\mu}}$ in the Θ-link (cf. §2) is subject to an unavoidable $\widehat{\mathbb{Z}}^{\times}$-indeter. "(Ind2)"

$$\widehat{\mathbb{Z}}^{\times} \curvearrowright \mathcal{O}_{\underline{\tilde{v}}}^{\times\boldsymbol{\mu}}$$

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$$\widehat{\mathbb{Z}}^{\times} \curvearrowright \mathcal{O}^{\times \boldsymbol{\mu}}_{\underline{\tilde{v}}}$$

... we shall refer to the compatibility/incompatibility — i.e., the functorial equivariance/nonfunctoriality — of a given Kummer theory with the "inter-universality indeterminacies" (Ind1), (Ind2) as the

multiradiality/uniradiality

of the Kummer theory; thus, the multiradiality of the Kummer theory may be understood as a sort of "splitting/decoupling" of the Kummer theory from the unit group $\mathcal{O}^{\times \boldsymbol{\mu}}_{\underline{\tilde{v}}}$

- Another Substantive Issue for Cyclotomic Rigidity Isomorphisms:

compatibility with the profinite/tempered topology, i.e., the property of admitting finitely truncated versions

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. . . this will be important since ring structures — which are necess. in order to define the power series for the p-adic log. (cf. $\mathfrak{log}$-link!) — only exist at "finite n" (cf. [Alien], §3.6, (ii); [EssLgc], Examples 3.8.3, 3.8.4), i.e., infinite "multiplicative Kumm. towers $\varprojlim_n$" destroy additive strs.!

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• In the case of the three types (a), (b), (c) of Kummer theory that are actually used in IUT (cf., especially, [Alien], Fig. 3.10; [Alien], §3.4, (v)):

(a) this approach to construct. CRI's is manifestly compat. w/ the prof. top., but is uniradial since it depends in an essential way on the ext. of $G_{\underline{v}}$-modules $1 \to \mathcal{O}^{\times}_{\underline{\tilde{v}}} \to K^{\times}_{\underline{\tilde{v}}} \to \mathbb{Q} \to 1$, hence is fundamentally incompat. w/ indet. $\widehat{\mathbb{Z}}^{\times} \curvearrowright \mathcal{O}^{\times}_{\underline{\tilde{v}}} \twoheadrightarrow \mathcal{O}^{\times\boldsymbol{\mu}}_{\underline{\tilde{v}}}$ (cf. [Alien], §3.4, (i));

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(b) it follows from the theory of the étale theta function — in particular, the symmetries of theta groups, together w/ the canonical splittings arising from restriction to 2- (or, alternatively, 6-) torsion points — that this approach to construct. CRI's is both compat. w/ the prof./tempered top. and multiradial (cf. [Alien], §3.4, (iii), (iv));

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(c) it follows from elementary considerations on "κ-coric" alg. rat. funct. that this approach to construct. CRI's is multiradial, but incompat. w/ the prof. top. (cf. [Alien], Example 2.13.1; [Alien], §3.4, (ii))

• The indet. $\widehat{\mathbb{Z}}^\times \curvearrowright \mathcal{O}^\times_{\underline{\tilde{v}}} \twoheadrightarrow \mathcal{O}^{\times\boldsymbol{\mu}}_{\underline{\tilde{v}}}$ of (a) means that the theta values and elements $\in F_{\mathrm{mod}}$ obtained by Galois evaluation

$$\left(\begin{array}{c}\text{Kummer class of some}\\ \text{sort of function}\end{array}\right)\Bigg|_{\text{decomposition group of a point}}$$

in (b), (c) are only meaningful — i.e., can only be protected from the $\widehat{\mathbb{Z}}^\times$-indet. — if they are considered, by applying the "non-interference" (up to roots of unity) of the monoids of (a) w/ those of (b) and (c), in terms of their actions on log-shells

$$\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j=1,\dots,l^*} \curvearrowright \mathcal{I}_{\underline{v}} \stackrel{\mathrm{def}}{=} \frac{1}{2p_{\underline{v}}}\log_{p_{\underline{v}}}(\mathcal{O}^{\times\boldsymbol{\mu}}_{\underline{v}}) \curvearrowleft F^\times_{\mathrm{mod}}$$

... whose def requires one to apply the $p_{\underline{v}}$-adic log., i.e., the $\mathfrak{log}$-link vert. shift. by -1, rel. to the coordine. of the Hodge theater that gave rise to the theta vals. and elts. $\in F_{\mathrm{mod}}$ under consider. (cf. [Alien], §3.7, (i)).

• Here, we recall that only the multiplicat. monoid $\mathcal{O}_{\underline{v}}^{\times\boldsymbol{\mu}}$ — i.e., not the ring strs., $\mathfrak{log}$-link, etc.! — is accessible, via the common data (cf. "$\wedge$!") in the gluing of the Θ-link, to the opposite side of the Θ-link!

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$\Longrightarrow$ to overcome the vertical $\mathfrak{log}$-shift discussed above, it is necessary to construct invariants w.r.t. the $\mathfrak{log}$-link (cf. §2!).

Here, we recall that ét-like strs. "$\circ$" (e.g., "$\Pi_{\underline{v}}$") are indeed $\mathfrak{log}$-link-inv., but the diagram — called the $\mathfrak{log}$-Kummer correspondence — arising from the vertical column (written horizontally) in the dom. of the Θ-link

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{\mathfrak{log}} & \bullet & \xrightarrow{\mathfrak{log}} & \bullet & \xrightarrow{\mathfrak{log}} & \bullet & \xrightarrow{\mathfrak{log}} & \dots \\
 & \dots & & \searrow & \downarrow & \swarrow & & \dots & \\
 & & & & \circ & & & &
\end{array}$$

— where the vert./diag. arrows are Kum. isoms. — is not commutative!

On the other hand, it is upper semi-commutative (!), i.e., all composites of Kummer and $\mathfrak{log}$-link morphisms on $\mathcal{O}_{\underline{v}}^{\times}$

$$\mathcal{O}_{\underline{v}}^{\times} \hookrightarrow \mathcal{O}_{\underline{v}} \hookrightarrow \mathcal{I}_{\underline{v}} \hookleftarrow \log_{p_{\underline{v}}}(\mathcal{O}_{\underline{v}}^{\times \boldsymbol{\mu}})$$

have images $\subseteq$ log-shell $\mathcal{I}_{\underline{v}}$ (cf. [Alien], Example 2.12.3, (iv)).

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It is (Ind3) that gives rise, ultimately, to the upper bound in the height inequalities that are obtained in IUT (cf. [Alien], Example 2.12.3, (iv); [Alien], §3.6, (iv); [Alien], §3.7, (i), (ii)).

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- In summary, we have two Kummer-detachment indeterminacies, namely,

(Ind2), (Ind3).

§5. Conjugate synchronization and the structure of ($\Theta^{\pm \mathrm{ell}}$NF-) Hodge theaters

- Fundamental Question: So how do we "simulate" GMS + GCG?

• In a word, we consider certain finite étale coverings over $K = F(E[l])$ of

$X \overset{\text{def}}{=} E \setminus \{\text{origin}\}, \quad C \overset{\text{def}}{=} X//\{\pm 1\}$ ("//": "stack-theoretic quotient")

determined by some rank one quotient $E[l]_K \twoheadrightarrow Q$:

$$\underline{X}_K \to X_K \overset{\text{def}}{=} X \times_F K \quad \text{...determined by } E[l]_K \twoheadrightarrow Q$$

$$\underline{C}_K \to C_K \overset{\text{def}}{=} C \times_F K \quad \text{...by taking } \underline{C}_K \overset{\text{def}}{=} \underline{X}_K//\{\pm 1\}$$

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and restr. to "local analytic sections" of $\mathrm{Spec}(K) \to \mathrm{Spec}(F)$ — called "prime-strips" (of which there are var. types — cf. [IUTchI], Fig. I1.2), which may be thought of as a sort of monoid- or Gal.-theoretic version of adèles/idèles — det'd by various $\mathrm{Gal}(K/F)$-orbits of the subset/section

$$\mathbb{V}(K) \supseteq \underline{\mathbb{V}} \stackrel{\sim}{\to} \mathbb{V}_{\mathrm{mod}} \stackrel{\text{def}}{=} \mathbb{V}(F_{\mathrm{mod}})$$

where $\mathrm{Ker}(E[l]_K \twoheadrightarrow Q)$ is the simulated "multipl. subsp.", or where some generator, up to ± 1, of Q is indeed the "canonical generator".

Note: Working w/ such prime-strips means that many conventional objects assoc. to number fields — such as abs. global Gal. gps or prime decomposition trees — must be abandoned!

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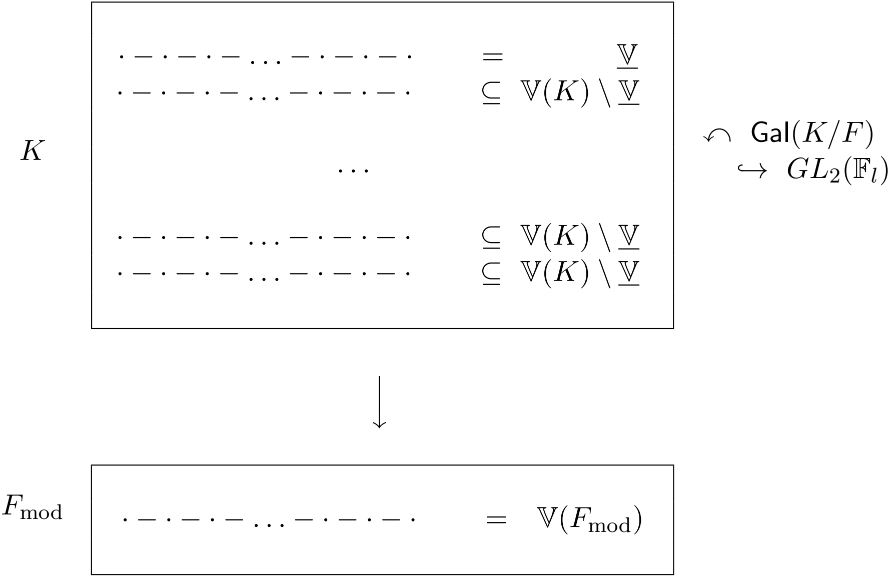
$\Longrightarrow$ need to apply the p-adic absolute mono-anabelian geometry of [AbsTopIII], §1 [cf. [Alien], §3.3, (iv)]!


= $\underline{\mathbb{V}}$
⊆ $\mathbb{V}(K) \setminus \underline{\mathbb{V}}$
K
$\curvearrowleft$ Gal(K/F)
$\hookrightarrow GL_2(\mathbb{F}_l)$
...
⊆ $\mathbb{V}(K) \setminus \underline{\mathbb{V}}$
⊆ $\mathbb{V}(K) \setminus \underline{\mathbb{V}}$
F_{mod}
= $\mathbb{V}(F_{\mathrm{mod}})$

- The hyperbolic orbicurves $\underline{X}_K$, $\underline{C}_K$ admit symmetries

$$\mathbb{F}_l^{\rtimes\pm} \overset{\text{def}}{=} \mathbb{F}_l \rtimes \{\pm 1\} \hookrightarrow \mathrm{Aut}_K(\underline{X}_K) \subseteq \mathrm{Aut}(\underline{X}_K) \quad \ldots \text{additive/geometric!}$$

$$\mathrm{Aut}(\underline{C}_K) \hookrightarrow \mathsf{Gal}(K/F_{\mathrm{mod}}) \twoheadrightarrow \mathbb{F}_l^{*} \overset{\text{def}}{=} \mathbb{F}_l^{\times}/\{\pm 1\} \quad \ldots \text{multiplicat./arithmetic!}$$

obtained by considering the resp. actions on cusps of $\underline{X}_K$, $\underline{C}_K$ that arise from elements of the quotient $E[l]_K \twoheadrightarrow Q$ [cf. [Alien], §3.3, (v); [Alien], §3.6, (i)].

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obtained by considering the resp. actions on cusps of $\underline{X}_K$, $\underline{C}_K$ that arise from elements of the quotient $E[l]_K \twoheadrightarrow Q$ [cf. [Alien], §3.3, (v); [Alien], §3.6, (i)]. At the level of arithmetic fundamental gps, these symmetries may be thought of as finite groups of outer automorphisms of

$$\Pi_{\underline{X}_K}, \quad \Pi_{\underline{C}_K}$$

where note that since, as is well-known, both the geom. fund. gp $\Delta_{\underline{X}_K}$ and the global abs. Gal. gp G_K are slim and do not admit fin. subgps of order > 2, these fin. gps of outer automorphisms do not lift to fin. gps of (non-outer) automorphisms (cf. [EssLgc], Example 3.8.2)!

(multiplicative/arithmetic) $\mathbb{F}_l^{\circledast}$-symmetries

Note: Since it is of crucial importance to fix the quot $E[l]_K \twoheadrightarrow Q$ by the "simulated GMS", we want to start from $\underline{C}_K$ and descend, via the multiplicative $\mathbb{F}_l^{\circledast}$-symmetries, to $C_{F_{\mathrm{mod}}}$ (where $C_{F_{\mathrm{mod}}} \times_{F_{\mathrm{mod}}} F = C$), not the other way around, which would obligate us to consider all Galois-, hence, in particular, all $SL_2(\mathbb{F}_l)$-conjugates of Q.

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... precisely the reverse (!) order to proceed from the point of view of classical Galois theory (cf. [Alien], §3.6, (iii); [EssLgc], Ex. 3.8.2).

In particular, the "strictly outer" nature of the multiplicative/arithmetic $\mathbb{F}_l^{\circledast}$-symmetries means that various copies of the absolute local Galois groups "$G_{\underline{v}}$" (for, say, nonarch. $\underline{v} \in \underline{\mathbb{V}}$) in the prime-strips that are permuted by these symmetries can only be identified with one another up to indeterminate inner automorphisms, i.e., there is no way to synchronize these conjugate indeterminacies (cf. [Alien], §3.6, (iii); [EssLgc], Example 3.8.2).

(additive/geometric) $\mathbb{F}_l^{\rtimes\pm}$-symmetries

By contrast, the "$G_{\underline{v}} \curvearrowright \mathcal{O}_{\underline{\tilde{v}}}^{\times\boldsymbol{\mu}}$" that appears in the gluing data for the Θ-link (cf. §2) must be indep. of the "$j \in \mathbb{F}_l^{*}$" (cf. the "$\underline{\underline{q}}^{j^2}$" of §2).

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That is to say, we need a "conjugate synchronized" $G_{\underline{v}}$ in order to construct the Θ-link, i.e., ultimately, in order to express the LHS of the Θ-link in terms of the RHS!!

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$\Longrightarrow$ This is done by applying the additive/geometric $\mathbb{F}_l^{\rtimes\pm}$-symmetries (cf. [Alien], §3.6, (ii); [EssLgc], Examples 3.8.2, 3.8.3, 3.8.4)!

Moreover, these additive/geometric $\mathbb{F}_l^{\rtimes\pm}$-symms. are compatible, rel. to the $\mathfrak{log}$-link, with the crucial local CRI's/Galois eval. of (a), (b) (but of (c) only up to conj. indets.! — cf. the $\mathbb{F}_l^{\circledast}$-symm. nature of (c) vs. the non-$\mathbb{F}_l^{\rtimes\pm}$-symm. nature of (b)!) of §4

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... precisely because these local CRI's of (a), (b) are compatible w/ the profinite/tempered topology, i.e., may be computed at a finite truncated level, where the ring str., hence also the power series for the p-adic logarithm, is well-defined (cf. [Alien], §3.6, (ii); [EssLgc], Examples 3.8.3, 3.8.4).

Note: This crucial property of compati'ty w/ the prof./temp. top. in the case of (b), as oppos. to (c), may be understood as a conseq. of the fact that the orders of the zeroes/poles at cusps of the theta fct. are all $= 1$!

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Moreover, this phenomenon may in turn be understood as a conseq. of the symmetries of theta gps, or, alterna'ly, as a conseq. of the quadrat. form/first Chern class "$\square^2$" in the exponent of the classical series rep'n of the theta fct. (cf. [Alien], §3.4, (iii), as well as the discussion below).

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By contrast, in the case of (c), the orders of the zeroes/poles at cusps of the algebraic rational functions that are used differ from one another by arbitrary elements of $\mathbb{Z} \setminus \{0\}$ (cf. [Alien], §3.4, (ii))!

$$\begin{bmatrix} -l^{*} & < & \dots & < & -1 & < 0 \\ & < & 1 & < & \dots & < & l^{*} \end{bmatrix} \qquad \qquad \begin{bmatrix} 1 & < & \dots \\ & < & l^{*} \end{bmatrix}$$

$$\Uparrow \qquad \Rightarrow \quad \text{glue!} \quad \Leftarrow \qquad \Uparrow$$

$$\underset{\curvearrowright}{\{\pm 1\}} \quad \begin{pmatrix} -l^{*} & < & \dots & < & -1 & < 0 \\ & < & 1 & < & \dots & < & l^{*} \end{pmatrix} \qquad \qquad \begin{pmatrix} 1 & < & \dots \\ & < & l^{*} \end{pmatrix}$$

$$\Downarrow \qquad \qquad \qquad \Downarrow$$

$$\begin{array}{ccc} \pm & \rightarrow & \pm \\ & \mathbb{F}_l^{\rtimes \pm} & \\ \uparrow & \curvearrowright & \downarrow \\ \pm & \leftarrow & \pm \end{array} \qquad \qquad \begin{array}{ccc} * & \rightarrow & * \\ & \mathbb{F}_l^{*} & \\ \uparrow & \curvearrowright & \downarrow \\ * & \leftarrow & * \end{array}$$

…additive, geometric symmetries

…multiplicative, arith. symmetries

• The properties of theta fcts. in IUT discussed above are particularly remarkable when viewed from the point of view of the analogy w/ the Jacobi identity for the theta fct. on the upper half-plane (cf. [EssLgc], Example 3.3.2; [ClsIUT], §4).

• The properties of theta fcts. in IUT discussed above are particularly remarkable when viewed from the point of view of the analogy w/ the Jacobi identity for the theta fct. on the upper half-plane (cf. [EssLgc], Example 3.3.2; [ClsIUT], §4). Indeed, on the one hand, the quadratic form/first Chern class "$\Box^2$" in the exponent of the classical series rep'n

$$\theta(t) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 t}$$

gives rise to the theta gp symmetries that underlie the rigidity properties of theta fcts. that play a central role in IUT from the point of view of the ultimate goal in IUT of expressing the LHS of the Θ-link in terms of the RHS — i.e., expressing the "Θ-pilot" on the LHS of the Θ-link in terms of the "q-pilot" on the RHS of the Θ-link.

On the other hand, this same quadratic form in the exponent of the classical series representation of the theta function

— which in fact appears as "$t \cdot \Box^2$", i.e., with a factor t, where t denotes coordinate on the imaginary axis of the upper half-plane

— also underlies the well-known Fourier transform invariance of the Gaussian distribution, up to a sort of "rescaling"

$$t \cdot \Box^2 \quad \mapsto \quad t^{-1} \cdot \Box^2.$$

It is precisely this rescaling that gives rise to the Jacobi identity.

This state of affairs is remarkable (cf. [ClsIUT], §3, §4) in that the trans. $t \mapsto t^{-1}$ corresponds to the linear fractional trans. given by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which, from the point of view of the analogy between the "infinite H" discussed at the end of §2 and the well-known bijection

$$\begin{array}{ccc} \mathbb{C}^\times \backslash GL_2^+(\mathbb{R})/\mathbb{C}^\times & \stackrel{\sim}{\to} & [0,1) \\ \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} & \mapsto & \frac{\lambda-1}{\lambda+1} \end{array}$$

(where $\lambda \in \mathbb{R}_{\geq 1}$), may be understood as follows:

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(where $\lambda \in \mathbb{R}_{\geq 1}$), may be understood as follows:

$$\left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right) \quad \longleftrightarrow \quad \Theta\text{-link} \quad \ldots \text{cf. "not } \Theta\text{-link-invariants"!}$$

$$\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \quad \longleftrightarrow \quad \mathfrak{log}\text{-link} \quad \ldots \text{cf. "}\mathfrak{log}\text{-link-invariants"!}$$

(cf. [Alien], §3.3, (ii); [EssLgc], §3.3, (InfH), Example 3.3.2).

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... in order to secure the l-torsion points at which one conducts the Galois evaluation of the étale theta function, i.e., the Kummer class of the (reciprocal of the l-th root of the) p-adic theta function (cf. the discussion of the Θ-link in §2; §4, (b))

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$$\underline{\underline{\Theta}}|_{l\text{-torsion points}} = \{\underline{\underline{q}}^{j^2}\}_{j=1,\dots,l^*}$$

... cf. the classical series representation of the theta function on the (imag. axis of the) upper half-plane — i.e., in essence, "$q = e^{2\pi i(it)}$"!

$$\theta(t) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 t} = \sum_{n=-\infty}^{+\infty} q^{\frac{1}{2}n^2}$$

§6. Multiradial representation and holomorphic hull

- Fundamental Theme:

To express/describe the Θ-pilot on the LHS of the Θ-link in terms of the RHS of the Θ-link, while keeping the Θ-link itself fixed (!)

• For instance, the labels "j" in "$\{\underline{\underline{q}}^{j^2}\}_{j=1,\dots,l^*}$" dep. on the complicated bookkeep. system for these essen'ly cuspidal labels (i.e., labels of cuspidal inertia groups in the geom. fundamental groups $\Delta_{\underline{v}} \stackrel{\text{def}}{=} \text{Ker}(\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}})$) furnished (cf. §5) by the structure of the Hodge theater on the LHS, which is not accessible from the point of view of the RHS.

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$\Longrightarrow$ necessary to express these labels in a way that is accessible from the RHS, i.e., by means of processions of capsules of prime-strips "/"

$$/ \;\hookrightarrow\; /\,/ \;\hookrightarrow\; /\,/\,/ \;\hookrightarrow\; \ldots \;\hookrightarrow\; /\ldots/$$

(i.e., successive inclusions of unordered collections of prime-strips of incrementally increasing cardinality) — which still yield symmetries between the prime-strips at different labels without "label-crushing", i.e., identifications between distinct labels (cf. [Alien], §3.6, (v)).

We then consider the *actions* of (b), (c) (cf. §4) on tensor-packets of the log-shells arising from the data of (a) (cf. §4) inside each capsule:

$$\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j=1,\ldots,l^*} \quad \curvearrowright \quad \mathcal{I}_{\underline{v}} \otimes \ldots \otimes \mathcal{I}_{\underline{v}} \quad \curvearrowleft \quad (F^{\times}_{\mathrm{mod}})_j$$

— where the "tensor-packet" is a tensor product of $j+1$ copies of $\mathcal{I}_{\underline{v}}$.

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$$\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j=1,\dots,l^*} \curvearrowright \mathcal{I}_{\underline{v}} \otimes \dots \otimes \mathcal{I}_{\underline{v}} \curvearrowleft (F_{\mathrm{mod}}^{\times})_j$$

— where the "tensor-packet" is a tensor product of $j+1$ copies of $\mathcal{I}_{\underline{v}}$.

• In fact, the various monoids, Galois groups, etc. that appear in the data (a), (b), (c) of §4 — such as $\mathcal{I}_{\underline{v}}$, $\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j=1,\dots,l^*}$, $(F_{\mathrm{mod}}^{\times})_j$, etc.
— come in four types (cf. [Alien], §3.6, (iv); [Alien], §3.7, (i)):

holomorphic Frobenius-like "(n, m)": monoids etc. on which $\Pi_{\underline{v}} \curvearrowright$ acts, and whose construction involves the ring structure associated to the Hodge theater at $(n, m) \in \mathbb{Z} \times \mathbb{Z}$;

holomorphic Frobenius-like "(n, m)": monoids etc. on which $\Pi_{\underline{v}} \curvearrowright$ acts, and whose construction involves the ring structure associated to the Hodge theater at $(n, m) \in \mathbb{Z} \times \mathbb{Z}$;

holomorphic étale-like "$(n, \circ)$": similar data to (n, m), but reconstructed from $\Pi_{\underline{v}}$, hence independent of "m";

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holomorphic étale-like "$(n, \circ)$": similar data to (n, m), but reconstructed from $\Pi_{\underline{v}}$, hence independent of "m";

mono-analytic Frobenius-like "$(n, m)^{\vdash}$": monoids, etc., on which $G_{\underline{v}} \curvearrowright$ acts; used in the gluing data — called an $\mathcal{F}^{\Vdash \blacktriangleright \times \boldsymbol{\mu}}$-prime-strip — that appears in the Θ-link;

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mono-analytic étale-like "$(n, \circ)^{\vdash}$": similar data to $(n, m)^{\vdash}$, but reconstructed from $G_{\underline{v}}$, hence independent of "m" (and in fact also of "n").

• Thus, in summary, the $\mathfrak{log}$-Kummer correspondence yields actions of the monoids of (b), (c) (cf. §4) on tensor-packets of log-shells arising from the data of (a) (cf. §4) up to the (Ind3)

$$\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j=1,\dots,l^*} \curvearrowright \mathcal{I}_{\underline{v}} \otimes \dots \otimes \mathcal{I}_{\underline{v}} \curvearrowleft (F_{\mathrm{mod}}^{\times})_j$$

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- first, at the level of objects of $(0, \circ)$;
- then by "descent" (i.e., the observ'n that reconst'ns from certain input data may in fact be conducted, up to natural isom., from less/weaker input data) up to (Ind1) at the level of objs. of $(0, \circ)^{\vdash}$;

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- first, at the level of objects of $(0, \circ)$;
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- then again by "descent" up to (Ind2) at the level of objs. of $(0, 0)^{\vdash} \stackrel{\sim}{\to} (1, 0)^{\vdash}$ (via the Θ-link).

$$(0,0) \overset{\text{(Ind3)}}{\rightsquigarrow} (0,\circ) \overset{\text{(Ind1)}}{\rightsquigarrow} (0,\circ)^{\vdash} \overset{\text{(Ind2)}}{\rightsquigarrow} (0,0)^{\vdash} \overset{\Theta\text{-link}}{\stackrel{\sim}{\to}} (1,0)^{\vdash}$$

(This last step involving (Ind2) plays the role of fixing the vertical coordinate, so that (Ind1), (Ind2) are not mixed with (Ind3) — cf. the discussion of "$\mathbb{C}^\times \backslash GL_2^+(\mathbb{R})/\mathbb{C}^\times$" at the end of §5!)

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This is the multiradial representation of the Θ-pilot on the LHS of the Θ-link in terms of the RHS (cf. [Alien], §3.7, (i); [EssLgc], §3.10, §3.11).

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This is the multiradial representation of the Θ-pilot on the LHS of the Θ-link in terms of the RHS (cf. [Alien], §3.7, (i); [EssLgc], §3.10, §3.11).

This multiradial representation plays the important role of exhibiting the (value group portion of the) Θ-pilot at $(0, 0)$ (i.e., which appears in the Θ-link!) as one of the possibilities within a container arising from the RHS of the Θ-link (cf. the "infinite H" at the end of §2; [EssLgc], §3.6, §3.10).

Next, by applying the operation of forming the holomorphic hull (i.e., "$\mathcal{O}_{\underline{v}}$-module generated by") to the various output regions of the multiradial representation, we obtain a module over the local $\mathcal{O}_{\underline{v}}$'s on the RHS of the Θ-link.

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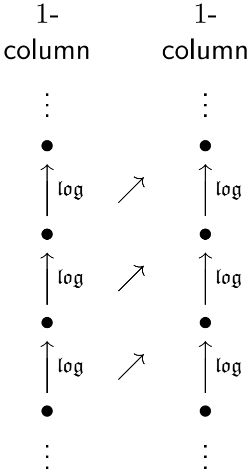
Then taking a suitable root of "det$(-)$" of this module yields an arithmetic line bundle — relative to the local $\mathcal{O}_{\underline{v}}$'s in the zero label! — in the same category as the category that gives rise to the q-pilot on the RHS of the Θ-link, except for a vertical log-shift by $+1$ in the 1-column (cf. the construction of log-shells from the "$\mathcal{O}_{\underline{\tilde{v}}}^{\times \boldsymbol{\mu}}$'s" that appear in the gluing data of the Θ-link!) — cf. [EssLgc], §3.10.

Thus, by symmetrizing (i.e., w. r. t. vertical shifts in the 1-column) the procedure described thus far, we obtain a closed loop, i.e.,



a situation in which the distinct labels on either side of the Θ-link (cf. the discussion at the beginning of §2!) may be eliminated, up to *suitable indeterminacies* (i.e., (Ind1), (Ind2), (Ind3); the holomorphic hull).

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In particular, by performing an entirely elementary log-volume computation, one obtains a nontrivial height inequality.
This completes the proof of the main theorems of IUT
(cf. [Alien], §3.7, (ii); [EssLgc], §3.10, §3.11).

Here, it is important to note that although the term “closed loop” at first might seem to suggest issues of “diagram commutativity” or “log-volume compatibility” — i.e., issues of

> “How does one conclude a relationship between the output data and the input data of the closed loop?”

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— in fact, such issues simply do not exist in this situation!

That is to say, the essential logical structure of the situation

$$
\begin{aligned}
A \wedge B \quad &= \quad A \wedge (B_1 \vee B_2 \vee \ldots) \\
&\Longrightarrow \quad A \wedge (B_1 \vee B_2 \vee \ldots \vee B_1' \vee B_2' \vee \ldots) \\
&\Longrightarrow \quad A \wedge (B_1 \vee B_2 \vee \ldots \vee B_1' \vee B_2' \vee \ldots \vee B_1'' \vee B_2'' \vee \ldots) \\
&\qquad \vdots
\end{aligned}
$$

proceeds by fixing the logical AND "$\wedge$" relation satisfied by the Θ-link and then adding various logical OR "$\vee$" indeterminacies, as illustrated in the following diagram (cf. [EssLgc], §3.10):

$$
\begin{array}{cccccccccccccccc}
 & & & & & & & & \bullet & = & = & \overset{\wedge}{=} & = & = & \bullet & \\
\rightsquigarrow & & & & & & (& \vee & \bullet & =) & = & \overset{\wedge}{=} & = & = & \bullet & \\
\rightsquigarrow & & & & & (& \vee & \vee & \bullet & = & =) & \overset{\wedge}{=} & = & = & \bullet & \\
\rightsquigarrow & & & & (& \vee & \vee & \vee & \bullet & = & = & \overset{\wedge}{=}) & = & = & \bullet & \\
\rightsquigarrow & & & (& \vee & \vee & \vee & \vee & \bullet & = & = & \overset{\wedge}{=} & =) & = & \bullet & \\
\rightsquigarrow & & (& \vee & \vee & \vee & \vee & \vee & \bullet & = & = & \overset{\wedge}{=} & = & =) & \bullet & \\
\rightsquigarrow & (& \vee & \vee & \vee & \vee & \vee & \vee & \bullet & = & = & \overset{\wedge}{=} & = & = & \bullet &)
\end{array}
$$